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Averaged controllability for random evolution Partial Differential Equations

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ABSTRACT

We analyze the averaged controllability properties of random evolution Partial Differential Equations. We mainly consider heat and Schrödinger equations with random parameters, although the problem is also formulated in an abstract frame. We show that the averages of parabolic equations lead to parabolic-like dynamics that enjoy the null-controllability properties of solutions of heat equations in an arbitrarily short time and from arbitrary measurable sets of positive measure. In the case of Schrödinger equations we show that, depending on the probability density governing the random parameter, the average may behave either as a conservative or a parabolic-like evolution, leading to controllability properties, in average, of very different kind.

RÉSUMÉ

On analyse la propriété de contrôlabilité en moyenne d'équations aux dérivées partielles dépendant de paramètres. On considére surtout les équations de la chaleur et de Schrödinger avec des paramètres aléatoires, bien que le problème soit également formulé dans un cadre abstrait. On montre que les moyennes des équations paraboliques conduisent à des dynamiques de nature parabolique qui vérifient la propriété de contrôlabilité à zéro, comme les solutions de l'équation de la chaleur, en un temps arbitrairememt court, à partir d'ensembles mesurables arbitraires de mesure positive. Dans le cas des équations de Schrödinger on montre que, en fonction de la densité de probabilité, la moyenne peut se comporter soit comme un modèle de type conservatif soit comme une équation parabolique, conduisant à des propriétés de contrôlabilité en moyenne très différentes.

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1. Introduction

We analyze the problem of controlling systems with randomly depending coefficients in the context of evolution Partial Differential Equations (PDEs). More precisely, we consider the problem of averaged controllability which consists, roughly, of controlling the averaged dynamics, with respect to the random parameters. This problem was introduced and solved in [54] in the context of finite dimensional systems, where the same issue was also formulated for PDE.

When the dynamics of the state is governed by a pair of random operators (determining the free dynamics and the control operators, respectively), generally speaking, controlling the system would require to know the actual value of the random parameters. But this is unfeasible. To avoid this paradox, controls should be independent of the unknown parameters. But, of course, this restricts our ability to deal with the randomness of the system. Accordingly the control requirement needs to be relaxed. The most natural relaxation is to require the control to perform optimally in an averaged sense. This amounts to controlling the mathematical expectation of the solutions, making a robust compromise of all the possible realizations of the system for the various possible values of the random parameters.

In this work we consider mainly the heat and Schrödinger equations, with diffusivity and dispersivity operators depending on a random variable in a multiplicative manner. We show that, while the average of heat equations leads also to a heat-like dynamics, the behavior of the averages for the Schrödinger equations depends in a very sensitive manner on the density of probability of the random variable so that, in some cases, by averaging, this leads to a dynamics of conservative nature, similar to the original Schrödinger equation under consideration and, in others, to a parabolic-like behavior.

Our method of proof combines using Fourier decomposition methods to identify the averaged dynamics to, later, utilizing the existing tools developed for the controllability of parabolic and conservative systems, to deduce the averaged controllability results. When the resulting averaged dynamics is of parabolic nature, averaged null controllability is proved for arbitrarily short time intervals and from measurable sets of positive measure. On the contrary, when the averaged dynamics is of conservative type, averaged exact controllability is proved under suitable geometric conditions on the support of the controls that are by now well known in the context of wave-like and Schrödinger-like equations.

In order to illustrate the effect of averaging and how it may change the dynamics of the original system, let us consider the simplest transport equation

$$\begin{cases} y_t + \alpha \cdot \nabla y = 0 & \text{in } \mathbb{R}^d \times [0, \infty), \\ y(0) = y_0 & \text{in } \mathbb{R}^d. \end{cases} \tag{1}$$

Here $y_0 \in L^2(\mathbb{R}^d)$ and $\alpha(\cdot) : \Omega \to \mathbb{R}^d$ is a d-dimensional standard normally distributed random variable, with the probability density

$$\rho(\alpha) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|\alpha|^2}{2}} \text{ for } \alpha \in \mathbb{R}^d.$$

The solution to (1) reads

$$y(x, t, \omega; y_0) = y_0(x - t\alpha) \text{ for } (x, t) \in \mathbb{R}^d \times [0, \infty).$$

Then, the mathematical expectation or averaged state