J. Math. Pures Appl. 104 (2015) 227–257

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Algebraic and combinatorial rank of divisors on finite graphs

Lucia Caporaso [a,*], Yoav Len [b], Margarida Melo [a,c]

[a] Department of Mathematics and Physics, Roma Tre University, Largo San L. Murialdo 1, 00146 Rome, Italy
[b] Mathematics Department, Yale University, 10 Hillhouse Ave, New Haven, CT 06511, USA
[c] CMUC and Mathematics Department of the University of Coimbra, Apartado 3008, EC Santa Cruz 3001-501 Coimbra, Portugal

ARTICLE INFO

Article history:
Received 2 April 2014
Available online 24 February 2015

MSC:
14H10
05C99
14H51

Keywords:
Graph
Algebraic curve
Divisor
Line bundle
Rank

ABSTRACT

We study the algebraic rank of a divisor on a graph, an invariant defined using divisors on algebraic curves dual to the graph. We prove it satisfies the Riemann–Roch formula, a specialization property, and the Clifford inequality. We prove that it is at most equal to the (usual) combinatorial rank, and that equality holds in many cases, though not in general.

RÉSUMÉ

On étudie le rang algébrique pour les diviseurs sur un graphe, un invariant défini par les diviseurs sur les courbes algébriques duales au graphe. On démontre le Théorème de Riemann–Roch, la proprieté de « spécialisation », et l'inégalité de Clifford. On démontre qu'il est inférieur ou égal au rang combinatoriel, avec égalité dans certains cas mais pas en général.

1. Introduction

Since recent years, a lively trend of research is studying the interplay between the combinatorial and algebro-geometric aspects of the theory of algebraic curves; this has led to interesting progress both in algebraic geometry and graph theory. The goal of this paper is to contribute to this progress by investigating the connection between the notion of combinatorial rank of divisors on graphs, and the notion of rank of Cartier divisors on an algebraic curve.

Loosely speaking, our main result is that the combinatorial rank of a divisor on a graph (a computer-computable quantity bounded above by the degree) is a fitting uniform upper bound on the dimension

* Corresponding author.
E-mail addresses: caporaso@mat.uniroma3.it (L. Caporaso), yoav.len@yale.edu (Y. Len), melo@mat.uniroma3.it, mmelo@mat.uc.pt (M. Melo).

http://dx.doi.org/10.1016/j.matpur.2015.02.006
0021-7824/

of linear series on curves (a hard to compute quantity, unbounded regardless of the degree). To be more precise, we need some context.

In the theory of algebraic curves, combinatorial aspects naturally appear when dealing with all curves simultaneously, as points of an algebraic variety. Indeed, a typical phenomenon in algebraic geometry is that the set of equivalence classes of varieties with given discrete invariants is itself an algebraic variety, whose geometric properties reflect those of the varieties it parametrizes. The case we should here keep in mind is the space $\overline{M_g}$, of all connected nodal curves of genus g up to stable equivalence; it is a complete variety containing, as a dense open subset, the space of isomorphism classes of smooth curves. $\overline{M_g}$ has been a central object of study for a long time, and it has been successfully used to study the geometry of algebraic curves. Several topics in this field, among which many open problems, concern projective realizations of abstract curves, i.e. the theory of line bundles (or Cartier divisors) and linear series.

A systematic study of these matters requires combinatorial methods to handle singular curves. Moreover, several questions are successfully answered by degeneration techniques (specializing a smooth curve to a singular one), where combinatorial aspects are essential; examples of this are the Griffiths–Harris proof of the Brill–Noether theorem, in [13], or the Kontsevich, and others, recursive formulas enumerating curves on surfaces, see [17] or [7].

In fact, since the first appearances of $\overline{M_g}$, as in the seminal paper [12], one sees associated to every nodal curve its *dual graph*, having as vertices the irreducible components of the curve, and as edges the nodes of the curve; moreover, every vertex of the dual graph is given a *weight*, equal to the geometric genus of the component it represents. For any (weighted) graph G we denote by $M^{\text{alg}}(G)$ the set of isomorphism classes of curves dual to G (i.e. having G as dual graph). Then we have

$$\overline{M_g} = (\sqcup M^{\text{alg}}(G))/\sim$$

where the union is over all connected graphs of genus g, and "$\sim$" denotes stable equivalence (which we don't define here, see [14]).

The dual graph is a key tool to deal with the combinatorial aspects mentioned above, especially in the theory of divisors and line bundles, when studying Néron models of Jacobians, Picard functors and compactified Jacobians, or degenerations of linear series.

More recently, and independently of the picture we just described, a purely combinatorial theory of divisors and linear series on graphs was being developed in a different framework; see [4] and [6]. The discovery that this graph-theoretic theory fits in well with the algebro-geometric set-up came somewhat as a surprise. To begin with, the group, $\text{Div}(G)$, of divisors on a graph G is the free abelian group on its vertices. The connection to line bundles on curves is simple: given a curve X dual to G, the multidegree of a line bundle on X is naturally a divisor on G, so we have a map $\text{Pic}(X) \to \text{Div}(G)$.

Such developments in graph-theory provide a fertile ground to extract and study the combinatorial aspects of the theory of algebraic curves; a remarkable example of this is the recent proof of the above mentioned Brill–Noether theorem, given in [11].

In this spirit, as mentioned at the outset, the goal of this paper is to interpret the combinatorial rank, $r_G(\underline{d})$, of a divisor, $\underline{d}$, on a graph G (as defined in [6] and in [2]) by the theory of algebraic curves. We do that by studying another invariant, the *algebraic rank*

$$r^{\text{alg}}(G, \underline{d})$$

of the divisor $\underline{d}$, defined in a completely different fashion, using algebro-geometric notions.

In algebraic geometry, the notion corresponding to the combinatorial rank is the rank of a line bundle, i.e. the dimension of the space of its global sections diminished by one. Now, the algebraic rank of a divisor on a graph G should be thought of as a uniform "sensible" upper bound on the rank of any line bundle