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On a quasilinear mean field equation with an exponential nonlinearity

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ABSTRACT

The mean field equation involving the N-Laplace operator and an exponential nonlinearity is considered in dimension $N \geq 2$ on bounded domains with homogeneous Dirichlet boundary condition. By a detailed asymptotic analysis we derive a quantization property in the non-compact case, yielding to the compactness of the solutions set in the so-called non-resonant regime. In such a regime, an existence result is then provided by a variational approach.

RÉSUMÉ

L'équation de champ moyen avec l'opérateur de N-Laplace et une non-linéarité exponentielle est considérée en dimension $N \geq 2$ sur des domaines bornés avec condition Dirichlet homogène au bord. Par une analyse asymptotique fine on dérive une propriété de quantification dans le cas non-compact, en produisant la compacité de l'ensemble des solutions dans le régime non-résonnant. Dans ce régime, un résultat d'existence est alors établi par une approche variationnelle.

1. Introduction

We are concerned with the following quasilinear mean field equation

$$\begin{cases} -\Delta_N u = \lambda \frac{V e^u}{\int_\Omega V e^u dx} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

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on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ denotes the N-Laplace operator, V is a smooth nonnegative function and $\lambda \in \mathbb{R}$. In the sequel, (1) will be referred to as the N-mean field equation.

In terms of λ or $\rho = \frac{\lambda}{\int Ve^u}$, the planar case $N = 2$ on Euclidean domains or on closed Riemannian surfaces has strongly attracted the mathematical interest, as it arises in conformal geometry [18,19,44], in statistical mechanics [16,17,20,46], in the study of turbulent Euler flows [29,64] and in connection with self-dual condensates for some Chern–Simons–Higgs model [25,28,32,37,51,52,58].

For $N = 2$ Brézis and Merle [15] initiated the study of the asymptotic behavior for solutions of (1) by providing a concentration-compactness result in Ω without requiring any boundary condition. A quantization property for concentration masses has been later given in [48], and a very refined asymptotic description has been achieved in [23,47]. A first natural question concerns the validity of a similar asymptotic behavior in the quasilinear case $N > 2$, where the nonlinearity of the differential operator creates an additional difficulty. The only available result is a concentration-compactness result [2,61], which provides a too weak compactness property towards existence issues for (1). Since a complete classification for the limiting problem

$$\begin{cases} -\Delta_N U = e^U \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} e^U < \infty \end{cases} \tag{2}$$

is not available for $N > 2$ (except for extremals of the corresponding Moser–Trudinger's inequality [43,50]) as opposite to the case $N = 2$ [21], the starting point of Li–Shafrir's analysis [48] fails and a general quantization property is completely missing. Under a "mild" control on the boundary values of u, Y.Y. Li and independently Wolanski have proposed for $N = 2$ an alternative approach based on Pohozaev identities, successfully applied also in other contexts [6,7,66]. The typical assumption on V is the following:

$$\frac{1}{C_0} \leq V(x) \leq C_0 \text{ and } |\nabla V(x)| \leq C_0, \qquad \forall\, x \in \Omega \tag{3}$$

for some $C_0 > 0$.

Pushing the analysis of [2,61] up to the boundary and making use of the above approach, our first main result is the following:

Theorem 1. *Let $u_k \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, be a sequence of weak solutions to*

$$-\Delta_N u_k = V_k e^{u_k} \qquad \text{in } \Omega, \tag{4}$$

where V_k satisfies (3) for all $k \in \mathbb{N}$. Assume that

$$\sup_{k\in\mathbb{N}} \int_\Omega e^{u_k} < +\infty \tag{5}$$

and

$$osc_{\partial\Omega} u_k = \sup_{\partial\Omega} u_k - \inf_{\partial\Omega} u_k \leq M$$

for some $M \in \mathbb{R}$. Then, up to a subsequence, u_k verifies one of the following alternatives: either

(i) u_k is uniformly bounded in $L^\infty_{loc}(\Omega)$ or
(ii) $u_k \to -\infty$ as $k \to +\infty$ uniformly in $L^\infty_{loc}(\Omega)$ or