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Global continuation beyond singularities on the boundary for a degenerate diffusive Hamilton–Jacobi equation

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ABSTRACT

In this article, we are interested in the Dirichlet problem for parabolic viscous Hamilton–Jacobi equations. It is well-known that the gradient of the solution may blow up in finite time on the boundary of the domain, preventing a classical extension of the solution past this singularity. This behavior comes from the fact that one cannot prescribe the Dirichlet boundary condition for all time and, in order to define a solution globally in time, one has to use "generalized boundary conditions" in the sense of viscosity solution. In this work, we treat the case when the diffusion operator is the p-Laplacian where the gradient dependence in the diffusion creates specific difficulties. In this framework, we obtain the existence and uniqueness of a continuous, global in time, viscosity solution. For this purpose, we prove a Strong Comparison Result between semi-continuous viscosity sub and super-solutions. Moreover, the asymptotic behavior of $\dfrac{u(x,t)}{t}$ is analyzed through the study of the associated ergodic problem.

RÉSUMÉ

On s'intéresse au problème de Dirichlet pour des équations paraboliques de type Hamilton–Jacobi avec diffusion non linéaire. Il est bien connu que le gradient de la solution peut exploser en temps fini sur le bord du domaine, ce qui constitue un obstacle pour étendre la solution au delà des singularités. Ce phénomène est lié à des pertes de conditions aux limites qui peuvent se produire à cause de la forte non-linéarité du terme hamiltonien et on doit utiliser la notion de condition aux limites généralisée au sens de la théorie des solutions de viscosité. Dans cet article on traite le cas du p-Laplacien où la dépendance en gradient dans la diffusion crée des difficultés spécifiques. On démontre un résultat de comparaison fort qui nous permet d'obtenir l'existence et l'unicité d'une solution de viscosité continue définie

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pour tous temps. De plus le comportement asymptotique de $\frac{u(x,t)}{t}$ est analysé via l'étude du problème ergodique associé.

1. Introduction and main results

In this article we are interested in the following generalized Dirichlet problem for second-order degenerate parabolic partial differential equations

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) + |Du|^q = f(x,t) \quad \text{in } \Omega \times (0,+\infty), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{on } \overline{\Omega}, \tag{1.2}$$

$$u(x,t) = g(x,t) \quad \text{on } \partial\Omega \times (0,+\infty), \tag{1.3}$$

where $q > p \geq 2$, u_0 and g are continuous functions satisfying the compatibility condition

$$u_0(x) = g(x,0) \quad \text{on } \partial\Omega. \tag{1.4}$$

Most of works devoted to this degenerate diffusive Hamilton–Jacobi equation concerned the case, where $\Omega = \mathbb{R}^N$, providing results on well-posedness, gradient estimates and asymptotic behavior of either classical or weak solutions in the sense of distributions (see [10,1,23,21] and the references therein).

Some other works are concerned with the solvability of the Cauchy–Dirichlet problem. They proved that, under suitable assumptions on u_0 and g, there exists a weak solution on some time interval $[0, T_{max}(u_0))$, with the property that its gradient blows up on the boundary $\partial\Omega$ while the solution itself remains bounded. We refer the reader to [2,18] and [22] for the degenerate parabolic case and to [25,24] for the uniformly parabolic case. This singularity is a difficulty to extend the solution past $T_{max}(u_0)$. A natural question is then: Can we extend the weak solution past $t = T_{max}(u_0)$ and in which sense?

Let us mention here that a result in this direction where the continuation beyond gradient blow-up does not satisfy the original boundary conditions was obtained in [15,16].

Recently, for the linear diffusion case ($p = 2$), Barles and Da Lio [6] showed that such gradient blow-up is related to a loss of boundary condition and address the problem through a viscosity solutions approach. They proved a "Strong Comparison Result" (that is a comparison result between *discontinuous* viscosity sub and supersolutions) which allowed them to obtain the existence of a unique continuous, *global in time viscosity solution* of (1.1)–(1.3), the Dirichlet boundary condition being understood in the *generalized sense of viscosity solution theory*. They also provided an explicit expression of the solution of (1.1)–(1.3) in terms of a value function of some exit time control problem, which allows a simple explanation of the losses of boundary condition when it arises.

We recall that the formulation of the generalized Dirichlet boundary condition for (1.1)–(1.3) in the viscosity sense reads

$$\min\left(u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) + |Du|^q - f(x,t), u - g\right) \leq 0 \quad \text{on } \partial\Omega \times (0,+\infty), \tag{1.5}$$

and

$$\max\left(u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) + |Du|^q - f(x,t), u - g\right) \geq 0 \quad \text{on } \partial\Omega \times (0,+\infty). \tag{1.6}$$

Our first result mainly extends the investigation of [6] to the degenerate diffusion case $p > 2$.