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Shell interactions for Dirac operators ☆

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ABSTRACT

The self-adjointness of $H + V$ is studied, where $H = -i\alpha \cdot \nabla + m\beta$ is the free Dirac operator in $\mathbb{R}^3$ and V is a measure-valued potential. The potentials V under consideration are given by singular measures with respect to the Lebesgue measure, with special attention to surface measures of bounded regular domains. The existence of non-trivial eigenfunctions with zero eigenvalue naturally appears in our approach, which is based on well known estimates for the trace operator defined on classical Sobolev spaces and some algebraic identities of the Cauchy operator associated to H.

RÉSUMÉ

On étudie le caractère auto-adjoint de $H + V$, où $H = -i\alpha \cdot \nabla + m\beta$ est l'opérateur de Dirac libre dans $\mathbb{R}^3$ et V est un potentiel à valeur mesure. Les potentiels V considerés sont donnés par des mesures singulières par rapport à la mesure de Lebesgue, avec une attention particulière au le cas des mesures de surface de domaines bornés réguliers. L'existence de fonctions propres non triviales à valeur propre nulle apparaît de façon naturelle dans notre approche, utilisant des estimations connues pour l'opérateur trace défini dans les espaces de Sobolev classiques et quelques identités algébriques de l'opérateur de Cauchy associé à H.

1. Introduction

In this article we investigate the self-adjointness in $L^2(\mathbb{R}^3)^4$ of the free Dirac operator

$$H = -i\alpha \cdot \nabla + m\beta \quad (\text{for } m > 0)$$

coupled with measure-valued potentials, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and α_j for $j = 1, 2, 3$ and β denote the so-called Dirac matrices (see (14) in Section 3 for the details about H). Recall that H acts on spinors $\varphi(x) = \binom{\phi}{\chi}(x)$ with $x \in \mathbb{R}^3$ and $\phi(x), \chi(x) \in \mathbb{C}^2$. Moreover, H is invariant under translations and, for $m = 0$, it is also invariant under scaling because, if

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$$\varphi_\lambda(x) = \lambda^{-1}\varphi(\lambda x) \quad \text{for } \lambda > 0,$$

then $H\varphi_\lambda(x) = H\varphi(\lambda x)$. We are interested on critical perturbations of H, i.e., those given by potentials $V(x)$ such that, when measured in an appropriate function space, the rescaled potentials

$$V_\lambda(x) = \lambda V(\lambda x) \quad \text{for } \lambda > 0 \tag{1}$$

also belong to the same space and have the same size. We shall pay special attention to potentials given by measures σ such that

$$\sigma(B) \leqslant C\,\mathrm{diam}(B)^2 \tag{2}$$

for any ball $B \subset \mathbb{R}^3$ (in particular, σ and the Lebesgue measure in $\mathbb{R}^3$ are mutually singular), and more precisely to surface measures of bounded regular domains. Note that, for balls centered at the origin, (2) is invariant under the scaling given by (1) in the distributional sense.

The main question that we want to address is the following: *in which sense these critical perturbations can be considered small with respect to the free Dirac operator H?* This can be seen as a very first step to understand more complicated settings, like for example those where V is given by a non-linear potential. At this respect it is worth mentioning that, as far as we know, all the available results concerning non-linear Dirac equations involve, in one way or another, some kind of smallness either on the size of the initial data or on the time of existence (see [9,14,3,4]).

The first kind of perturbation one could think about is the one given by potentials V that are hermitian and that grow like the classical Coulomb potential, that is

$$|x|\big|V(x)\big| \leqslant \nu \quad \text{for some } \nu \geqslant 0.$$

For $\nu < 1$, there exists a unique domain D where $H + V$ is self-adjoint and such that D is a subspace of the space of spinors with finite kinetic and potential energy, i.e.,

$$D \subset \left\{\varphi \in L^2\big(\mathbb{R}^3\big)^4\colon\ (I_4 - \Delta)^{1/4}(\varphi) \in L^2\big(\mathbb{R}^3\big)^4 \text{ and } \int |\varphi|^2 \frac{dx}{|x|} < +\infty\right\},$$

where I_4 denotes the identity operator on $L^2(\mathbb{R}^3)^4$ (see [12]). It is well known that, for $V(x) = \nu/|x|$ and $|\nu| > 1$, the hamiltonian is not essentially self-adjoint (see [19]), and it does not seem to exist a natural choice among all the possible extensions. Concerning the critical case $\nu = \pm 1$, little is known. For scalar potentials

$$V(x) = v(x)I_4 \quad \text{with } v(x) \in \mathbb{R},$$

partial results have been obtained in [8]. The existence of a threshold at $\nu = 1$ is a consequence of a sharp inequality of Hardy type that involves H instead of the usual gradient. Note that H does not leave invariant the set of radial spinors, hence this Hardy's inequality is not a straightforward extension of the classical one. Besides, recall that H is not a semibounded operator. In fact, assume that $V(x) = V(-x)$ and that $\varphi(x) = \binom{\phi}{\chi}(x)$ is an eigenfunction with eigenvalue λ. Then $\widetilde{\varphi}(x) = \binom{\chi}{\phi}(-x)$ is an eigenfunction with eigenvalue $-\lambda$. This elemental property plays a role in one of the main results in this paper, namely Theorem 3.8.

Motivated by the examples of potentials with Coulombic type singularities, we want to investigate the case of potentials with a singular support on a hypersurface $\Sigma \subset \mathbb{R}^3$; spheres and hyperplanes are fundamental examples. One may assume without loss of generality that the sphere is