J. Math. Pures Appl. 102 (2014) 672–701

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Structure of the positive solutions for supercritical elliptic equations in a ball

Yasuhito Miyamoto [1]

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

ARTICLE INFO

Article history:
Received 31 October 2012
Available online 14 February 2014

MSC:
35B32
35B33
35J61
35J25

Keywords:
Bifurcation diagram
Infinitely many turning points
Joseph–Lundgren exponent
Singular solution

ABSTRACT

Let $B \subset \mathbb{R}^N$ $(N \geqslant 3)$ be a unit ball. We consider the bifurcation diagram of the positive solutions to the supercritical elliptic equation in B

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u > 0 & \text{in } B, \end{cases}$$

where $f(u) = u^p + g(u)$ $(p > p_S := (N+2)/(N-2))$ and $g(u)$ is a lower order term which satisfies certain assumptions. We show that if $p_S < p < p_{JL}$, then the branch has infinitely many turning points around some $\lambda^* > 0$. Here,

$$p_{JL} := \begin{cases} 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}} & (N \geqslant 11), \\ \infty & (2 \leqslant N \leqslant 10). \end{cases}$$

We give an example such that the bifurcation diagram can be classified by using our theorem and the characterization of the extremal solution by Brezis and Vázquez. The main tool is the intersection number between regular and singular solutions.

RÉSUMÉ

Soit $B \subset \mathbb{R}^N$ $(N \geqslant 3)$ la boule unité. On considére le diagramme de bifurcation des solutions positives de l'équation elliptique supercritique dans B

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \\ u > 0 & \text{in } B, \end{cases}$$

où $f(u) = u^p + g(u)$ $(p > p_S := (N+2)/(N-2))$ et $g(u)$ est un terme d'ordre inférieur satisfaisant certaines conditions. On montre que, si $p_S < p < p_{JL}$, alors la branche a une infinité de points tournants autour d'un $\lambda^* > 0$. Ici,

$$p_{JL} = \begin{cases} 1 + \frac{4}{N-4-2\sqrt{N-1}} & (N \geqslant 11), \\ \infty & (2 \leqslant N \leqslant 10). \end{cases}$$

E-mail address: miyamoto@ms.u-tokyo.ac.jp.

[1] This work was partially supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (B) (Subject Nos. 21740116 and 24740100) and by Keio Gijuku Academic Development Funds.

http://dx.doi.org/10.1016/j.matpur.2014.02.002

On donne un exemple tel que le diagramme de bifurcation peut être classé à l'aide de notre théorème et de la caractérisation de la solution extrémale par Brezis et Vázquez. Le principal outil est le nombre d'intersections entre solutions régulières et singulières.

1. Introduction and main results

Let B be a unit ball in $\mathbb{R}^N$ ($N \geqslant 3$). In this paper we are interested in the global bifurcation diagram of the semilinear elliptic Dirichlet problem

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where

$$f(u) = u^p + g(u), \tag{1.2}$$

$$p > p_S := \frac{N+2}{N-2}, \tag{1.3}$$

$g(u)$ is a lower order term, and λ is a non-negative constant. The precise assumptions on f and g are given in (f1)–(f3) below. The exponent p_S is called the Sobolev critical exponent. Because $p > p_S$, the Sobolev embedding $H^1(B) \hookrightarrow L^{p+1}(B)$ does not hold. Hence, it is difficult to use a variational method in the function space $H^1(B)$. By the symmetry result of Gidas–Ni–Nirenberg [8], every positive solution u is radially symmetric and $\|u\|_\infty = u(0)$. This enables us to use ODE techniques. We assume $f(0) > 0$. Then there is an unbounded branch $\{(\lambda, u)\}$ consisting of positive radial solutions of (1.1) such that the branch emanates from $(0, 0)$.

Motivations of this study are the following two problems:

$$\begin{cases} \Delta u + \lambda(1+u)^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \Delta u + \lambda u + u^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{1.5}$$

Joseph–Lundgren [10] studied the positive radial branch of (1.4) and showed that (1.4) has two types of the solution structures depending on p and N. There exists $\bar{\lambda} > 0$ such that the solution to (1.4) exists only when $\lambda \leqslant \bar{\lambda}$. Let

$$p_{JL} := \begin{cases} 1 + \frac{4}{N-4-2\sqrt{N-1}} & (N \geqslant 11), \\ \infty & (2 \leqslant N \leqslant 10) \end{cases} \tag{1.6}$$

be the Joseph–Lundgren exponent introduced in [10]. When $p_S < p < p_{JL}$, there exists $\lambda^*(< \bar{\lambda})$ such that the branch emanating from $(0, 0)$ goes to $\lambda = \bar{\lambda}$ and bends back at $\bar{\lambda}$. This portion of the branch consists