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Compressible perturbation of Poiseuille type flow

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Abstract

The paper examines the issue of stability of Poiseuille type flows in regime of compressible Navier–Stokes equations in a three dimensional finite pipe-like domain. We prove the existence of stationary solutions with inhomogeneous Navier slip boundary conditions admitting nontrivial inflow condition in the vicinity of constructed generic flows. Our techniques are based on an application of a modification of the Lagrangian coordinates. Thanks to such an approach we are able to overcome difficulties coming from hyperbolicity of the continuity equation, constructing a maximal regularity estimate for a linearized system and applying the Banach fixed point theorem.

Résumé

Dans cet article on étudie la stabilité des mouvements de fluides de type Poiseuille dans le cadre des équations de Navier–Stokes compressibles dans un domaine cylindrique borné en trois dimensions. On montre l'existence de solutions stationnaires avec conditions au bord de type Navier non homogènes admettant la traversée du bord par le fluide, au voisinage d'une classe de solutions laminaires du type Poiseuille. Notre technique est fondée sur l'application de coordonnées de Lagrange modifiées. La methode permet de prendre en compte les difficultés résultant de l'hyperbolicité de l'équation de continuité en dérivant une estimée de régularité maximale pour un système linéarisé et en appliquant un théorème de point fixe de Banach.

MSC: 35Q30; 76N10

Keywords: Lagrangian coordinates; Navier–Stokes equations; Compressible flow; Slip boundary conditions; Inflow condition; Strong solutions; Maximal regularity

1. Introduction

The mathematical description of compressible flows is important from the point of view of applications, domains such as aerodynamics and geophysics are the most natural to be mentioned here. On the other hand, complexity of the equations describing the flow delivers very interesting mathematical challenges. In spite of active research in the field, we are still far from the complete mathematical understanding of compressible flows. The only general existence results are available for weak solutions with homogeneous boundary conditions [6,11]. As far as regular solutions are

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concerned, we have so far only partial results assuming either some smallness of the data, or its special structure. The problems have been investigated mainly with homogeneous boundary conditions [4,16]. For the overview of the state of art in the theory one can consult the monograph [17].

From the point of view of the aforementioned applications it seems very important to investigate the problems with large velocity vectors, which lead in a natural way to inhomogeneous boundary conditions. Due to the hyperbolic character of the continuity equation the density must be then prescribed on the inflow part of the boundary. Existence issues for such inflow problems are investigated in [8,9,18–21,25]. The mentioned group of problems can be regarded as questions of stability of particular constant flows.

In the present article we would like to examine the issue of stability of Poiseuille type flow in pipe-like domain in compressible regime. The Poiseuille flow is a special symmetric solution to the incompressible Navier–Stokes equations in cylindrical domains. Here it is viewed as a solution to the compressible Navier–Stokes system with constant density and constant external force, parallel to axis of the cylinder, given by the pressure (gravitation-like term). Hence the pressure, unknown in the incompressible model, is recognized as a given force. Such change of 'observer' looks acceptable from the mechanical point of view. Thanks to that interpretation we obtain a natural physically reasonable flow in compressible regime. The mathematical objective of this article is to establish stability of such flow under some structural assumptions limiting the magnitude of admissible perturbations.

The system. Let us define the system. We consider steady flow of a viscous, barotropic fluid in a bounded, cylindrical domain in $\mathbb{R}^3$, described by the Navier–Stokes system supplied with inhomogeneous Navier slip boundary conditions. The complete system reads

$$\begin{aligned}
&\rho v\cdot\nabla v-\mu\Delta v-(\mu+\nu)\nabla\operatorname{div}v+\nabla\pi(\rho)=\rho F && \text{in}\,\Omega,\\
&\operatorname{div}(\rho v)=0 && \text{in }\Omega,\\
&n\cdot\mathbf{T}(v,\pi)\cdot\tau_k+fv\cdot\tau_k=b_k,\quad k=1,2 && \text{on }\Gamma,\\
&n\cdot v=d && \text{on }\Gamma,\\
&\rho=\rho_{in} && \text{on }\Gamma_{in},
\end{aligned}\tag{1.1}$$

where $\Omega=[0,L]\times\Omega_0$ with $\Omega_0\subset\mathbb{R}^2$ of class C^2, Γ denotes the boundary of Ω (see Fig. 1), v is the velocity field of the fluid, ρ is its density, μ and ν are viscosity constants satisfying $\mu>0$ and $(\nu+2\mu)>0$, $f\geqslant 0$ is the friction coefficient which may be different on different components of the boundary Γ, $\pi=\pi(\rho)$ is the pressure given as a function, at least C^1, of the density and F is an external force. $\mathbf{T}$ denotes the Cauchy stress tensor of the form

$$\mathbf{T}(v,\pi)=2\mu\mathbf{D}(v)+\nu\operatorname{div}v\mathbf{Id}-\pi\mathbf{Id},\quad \text{where }\mathbf{D}(v)=\frac{1}{2}\bigl(\nabla v+\nabla v^T\bigr)$$

is the symmetric gradient. Next, n and τ_k are outer normal and tangent vectors to $\partial\Omega$. Boundary data ρ_{in}, b, d will be discussed later. The boundary Γ is naturally split into three parts:

$$\begin{aligned}
\Gamma_0&=\bigl\{x\in\partial\Omega:\ v(x)\cdot n(x)=0\bigr\},\\
\Gamma_{in}&=\bigl\{x\in\partial\Omega:\ v(x)\cdot n(x)<0\bigr\},\\
\Gamma_{out}&=\bigl\{x\in\partial\Omega:\ v(x)\cdot n(x)>0\bigr\}.
\end{aligned}\tag{1.2}$$

Thanks to the chosen geometry of the domain, the above decomposition is easily illustrated by Fig. 1.

We shall say few words about the physical interpretation of the system (1.1), in particular about the choice of boundary conditions $(1.1)_{3,4}$. We would like to model a flow through a pipe. We assume that the fluid obeys Navier slip conditions on the walls of the pipe (Γ_0 component of the boundary), hence natural conditions on Γ_0 are $d\equiv 0$ and $b_k\equiv 0$. However, the mathematical requirements impose a need to prescribe the boundary conditions on Γ_{in} and Γ_{out}. From the physical viewpoint these parts are artificial, this is the area where the parameters of the velocity and density are measured. This gives us a freedom of choice of the type of boundary conditions on the inflow and outflow part, which can be fit to the mathematical approach, hence we choose inhomogeneous slip condition. Note that as the friction coefficient goes to infinity, then the relations $(1.1)_{3,4}$, at least formally, become the standard Dirichlet conditions describing the whole velocity vector at the boundary. Since the velocity does not vanish on the boundary, the hyperbolicity of the continuity equation imposes a need to prescribe the density on the inflow part, which leads to