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Existence of critical points with semi-stiff boundary conditions for singular perturbation problems in simply connected planar domains

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Abstract

Let Ω be a smooth bounded simply connected domain in $\mathbb{R}^2$. We investigate the existence of critical points of the energy $E_\varepsilon(u) = 1/2 \int_\Omega |\nabla u|^2 + 1/(4\varepsilon^2) \int_\Omega (1 - |u|^2)^2$, where the complex map u has modulus one and prescribed degree d on the boundary. Under suitable nondegeneracy assumptions on Ω, we prove existence of critical points for small ε. More can be said when the prescribed degree equals one. First, we obtain existence of critical points in domains close to a disk. Next, we prove that critical points exist in "most" of the domains.

Résumé

Soit Ω un domaine borné lisse simplement connexe de $\mathbb{R}^2$. On édutdie l'existence de points critiques de l'énergie $E_\varepsilon(u) = 1/2 \int_\Omega |\nabla u|^2 + 1/(4\varepsilon^2) \int_\Omega (1 - |u|^2)^2$, où u est une fonction à valeurs complexes, dont la trace au bord est de module un et de degré prescrit d. Sous des hypothèses appropriées de non dégénérescence sur Ω, on démontre l'existence de points critiques pour ε petit. Dans le cas où le degré prescrit est égal à un, on peut préciser ce résultat. Premièrement, on obtient l'existence de points critiques dans tout domaine proche d'un disque. De plus, on démontre qu'il existe des points critiques dans la « plupart » des domaines.

Keywords: Ginzburg–Landau; Singular perturbations; Transversality

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply connected domain. Let a map u belong to the space

$$\mathcal{E} := \{u \in H^1(\Omega, \mathbb{C}); |\operatorname{tr} u| = 1\},$$

where $\operatorname{tr} u$ denotes the trace of u on the boundary $\partial\Omega$. Then the trace $\operatorname{tr} u$ of u on $\partial\Omega$ belongs to the space $H^{1/2}(\partial\Omega; \mathbb{S}^1)$, and therefore we can define its winding number or degree, which we denote by $\deg(u, \partial\Omega)$ (see [1, Appendix]; see also [2, Section 2] for more details). This allows us to define the class

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$$\mathcal{E}_d = \{u \in H^1(\Omega; \mathbb{C}); |\operatorname{tr} u| = 1, \deg(u, \partial\Omega) = d\}.$$

In this paper we study the existence of critical points of the Ginzburg–Landau energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1 - |u|^2)^2$$

in the space $\mathcal{E}_d$, i.e., of critical points with prescribed degree d. More specifically, we are interested in nontrivial critical points, that is critical points which are not constants of modulus one.

The prescribed degree boundary condition is an intermediate model between the Dirichlet and the Neumann boundary conditions. The asymptotic of minimizers of the Ginzburg–Landau energy E_ε with Dirichlet boundary condition was first studied by Bethuel, Brezis and Hélein in their classical work [3]. In particular, it was shown in [3] that minimizers u_ε have zeros "well-inside" Ω, and that these zeros approach the singularities (vortices) of the limit u_* of the u_ε's as $\varepsilon \to 0$. In contrast, the only minimizers of E_ε with no boundary condition are constants. The same holds even for stable critical points of E_ε with Neumann boundary conditions [4]. The analysis of the prescribed degrees boundary condition (in domains which may be multiply connected) leads to a richer global picture [5–10,2]. More specifically, in multiply connected domains minimizers of E_ε may exist [6,7] or not [8]. However, in such domains critical points of E_ε always exist [9,10]. In simply connected domains, minimizers never exist [7]. More involved is the study of the existence of critical points in simply connected domains; this is our purpose. Typical methods in absence of absolute minimizers consist in constructing local minimizers, or in constructing critical points by minimax methods. Construction of local minimizers proved to be successful in multiply connected domains [9], but the arguments there do not adapt to our case. Minimax techniques led in [2] to the proof of the existence of critical points in simply connected domains for *large* ε, but again these techniques do not seem to work for *small* ε.

The present paper is devoted to the existence of critical points for *small* ε and thus complements [2]. Our approach relies on singular perturbations techniques, in the spirit of Pacard and Rivière [11]. We explain this approach in the special case where the prescribed degree is $d = 1$. We first recall the main result in [3]. Consider the minimization of E_ε with Dirichlet boundary condition:

$$\min\{E_\varepsilon(u); \operatorname{tr} u = g \text{ on } \partial\Omega\}.$$

Here, $g : \partial\Omega \to \mathbb{S}^1$ is smooth, and we assume that $\deg(g, \partial\Omega) = 1$. Then there exists some $a \in \Omega$ such that, possibly up to a subsequence, minimizers u_ε satisfy $u_\varepsilon \to u_*$, with

$$u_*(z) = u_{*,a,g}(z) = \frac{z-a}{|z-a|}e^{\imath H}, \quad \text{with } H = H_{a,g} \text{ harmonic.} \tag{1.1}$$

In (1.1), the function H is uniquely determined (mod 2π) by the condition

$$u_* = g \quad \text{on } \partial\Omega. \tag{1.2}$$

The point a is not arbitrary: it has to be a critical point (actually, a point of minimum) of the "renormalized energy" $W(\cdot, g)$ associated with g.

In order to explain our main results in the case of prescribed degree boundary condition, we perform a handwaving analysis of our problem when $d = 1$. Assume that u_ε is a critical point of E_ε in $\mathcal{E}_1$. Then u_ε has to vanish at some point a_ε, and up to a subsequence we have either

(i) $a_\varepsilon \to a \in \Omega$,
or
(ii) $a_\varepsilon \to a \in \partial\Omega$.

Assume that (i) holds. Assume further, for the purpose of our discussion, that a_ε is the only zero of u_ε. Then the analysis in [3] suggests that the limit u_* of the u_ε's should be again of the form $u_*(z) = \frac{z-a}{|z-a|}e^{\imath\psi}$. Formally, the fact that u_ε is a critical point of E_ε leads, as in [3], to the conclusion that the limiting point a is a critical point of a suitable renormalized energy $\widehat{W}(\cdot)$. Some basic properties of the energy $\widehat{W}$ are studied in [12]; we will come back to this in Section 2. Of interest to us is the fact that $\widehat{W}$ is smooth and does have critical points.