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Homogenization of the oscillating Dirichlet boundary condition in general domains

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Abstract

We prove the homogenization of the Dirichlet problem for fully nonlinear uniformly elliptic operators with periodic oscillation in the operator and in the boundary condition for a general class of smooth bounded domains. This extends the previous results of Barles and Mironescu (2012) [4] in half spaces. We show that homogenization holds despite a possible lack of continuity in the homogenized boundary data. The proof is based on a comparison principle with partial Dirichlet boundary data which is of independent interest.

Résumé

Pour des opérateurs complètement non linéaires, uniformément elliptiques, à coefficients périodiques on considère le problème de l'homogénéisation pour des conditions au bord de Dirichlet également périodiques, pour un domaine borné à frontière réguliière. Les résultats généralisent ceux obtenus précédemment par Barles et Mironescu (2012) [4] dans un demi-espace. On montre que l'homogénéisation est possible, même dans le cas où les données au bord homogénéisées manquent de continuité, la démonstration utilise un principe de comparaison pour des données au bord partielles. La méthode présente un intérêt par elle-mème.

Keywords: Homogenization; Discontinuous boundary data; Fully nonlinear elliptic equations; Boundary layers

1. Introduction

We consider the homogenization of the following Dirichlet problem set in a bounded domain in $\Omega \subset \mathbb{R}^n$ with smooth boundary,

$$\begin{cases} F_\varepsilon\left(D^2u^\varepsilon, x, \dfrac{x}{\varepsilon}\right) = 0 & \text{for } x \in \Omega, \\ u^\varepsilon(x) = g\left(x, \dfrac{x}{\varepsilon}\right) & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

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with Hölder continuous boundary data $g \in C^\alpha(\mathbb{R}^n \times \mathbb{R}^n)$ for some $\alpha \in (0,1)$. The operator $(M,x,y) \to F_\varepsilon(M,x,y)$ is assumed to be uniformly elliptic, and both the operator and the boundary condition $g(x,y)$ are assumed to be $\mathbb{Z}^n$-periodic in the y variable. See Section 2.3 for the precise assumptions we make on the operators.

The homogenization of the problem (1.1) when $g(x,y) = g(x)$ is independent of the fast variable is somewhat classical at this point and was done by Evans [13,12], the ideas originated in the unpublished paper of Lions, Papanicolaou and Varadhan [20] on the periodic homogenization of Hamilton–Jacobi equations. There are too many works investigating the homogenization of the interior operators in various settings for us to list all of them here, we simply mention that more recently the interior homogenization has been shown in the very general setting of stationary ergodic random media by Caffarelli, Souganidis and Wang in [8].

As far as oscillating boundary data is concerned there is less literature. For divergence form operators the case of oscillating co-normal Neumann data is in the classical book of Bensoussan, Papanicolaou and Lions [6]. For the case of non-co-normal Neumann data and fully nonlinear operators there are a number of recent works. The papers of Arisawa [1] and Barles, Da Lio, Lions and Souganidis [3] identify the cell problem and show the homogenization in half-space type domains. The general domain case for the Neumann problem was solved by Choi and Kim in [9] (see also [10]). They require that the operator F_ε homogenizes to a rotationally invariant operator, in this situation the homogenized boundary data is continuous which guarantees the uniqueness of the limiting problem and thus stability of the ε-problem as $\varepsilon \to 0$. This issue is explained in more detail in [9]. Our work is based on the ideas in [9,10]. Indeed, one of the novelties of this article is to show the uniqueness and stability result for a general class of operators even though the homogenized boundary data may be discontinuous [17]. In particular we do not need to put any special assumptions on the operators besides the uniform ellipticity. We believe that similar ideas should apply to the case of oscillatory Neumann data.

For the oscillating Dirichlet data case there are even less references. For linear divergence form equations the homogenization in general domains was shown recently by Gérard-Varet and Masmoudi in [14,15] where they are able to consider linear systems, see also [19]. The work of Barles and Mironescu [4] deals with fully nonlinear non-divergence form elliptic operators when Ω is a half space. Barles and Mironescu identify the cell problem and the attendant difficulties in solving it under quite general assumptions on the operators. We will be concerned with extending this result to general domains. In order to clarify the main issues that arise from the general domain rather than from the cell problem we will avoid giving the most general assumptions on the operators. We expect that our similar methods should also work in the generality at which [4] show the cell problem can be solved, for more details see Remark 1.7.

At least on the surface a difficulty of this problem, especially in contrast to the corresponding oscillating Neumann data problem, is a lack of equicontinuity for the u^ε. The optimal estimate for the continuity of the u^ε up to the boundary cannot be any better than the continuity of $g(x, \frac{x}{\varepsilon})$ which has oscillations of size 1 at arbitrarily small scales in the limit $\varepsilon \to 0$. In particular the homogenization of (1.1), if it occurs, must be happening outside of a 'boundary layer' of width $o(1)$. Keeping this in mind, let us call $\overline{F}$ to be the homogenized operator on the interior given by the results of [13,12,8]. Our goal is to show that the solutions u^ε of the ε-problem (1.1) converge locally uniformly in Ω to $\bar{u}$ solving an equation of the form,

$$\begin{cases} \overline{F}(D^2\bar{u}, x) = 0 & \text{for } x \in \Omega, \\ \bar{u}(x) = \bar{g}(x) & \text{for } x \in \partial\Omega. \end{cases} \tag{1.2}$$

As we will see, the homogenized boundary condition $\bar{g}$ will depend not only on the function g, but also on ν_x, the inner normal to Ω at x and on the F_ε.

Let us describe heuristically the approach taken in [4] to identifying the homogenized boundary condition. The analysis of the problem (1.1) proceeds by blowing up about points $x \in \partial\Omega$. As in [4], the introduction of the localizations,

$$v^\varepsilon(y) = u^\varepsilon(x_0 + \varepsilon y),$$

at points $x_0 \in \partial\Omega$ in the limit as ε is taken to 0 leads us formally to the following "cell problem" set in the half space $P(x_0,\nu) := \{y: (y - x_0) \cdot \nu > 0\}$ where $\nu \in S^{n-1} = \{x \in \mathbb{R}^n: |x| = 1\}$,

$$\begin{cases} F(D^2 v, y) = 0 & \text{in } P(x_0,\nu), \\ v(y) = \psi(y) & \text{on } \partial P(x_0,\nu). \end{cases} \tag{1.3}$$