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Hybrid resonance of Maxwell's equations in slab geometry ☆

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Abstract

Hybrid resonance is a physical mechanism for the heating of a magnetic plasma. In our context hybrid resonance is a solution of the time harmonic Maxwell's equations with smooth coefficients, where the dielectric tensor is a non-diagonal Hermitian matrix. The main part of this work is dedicated to the construction and analysis of a mathematical solution of the hybrid resonance with the limit absorption principle. We prove that the limit solution is singular: it consists of a Dirac mass at the origin plus a principal value and a smooth square integrable function. The formula obtained for the plasma heating is directly related to the singularity.

Résumé

La résonnance hybride est un mécanisme physique qui permet de chauffer un plasma magnétique. Dans le contexte de ce travail, la résonnance hybride est une solution particulière des équations de Maxwell en régime harmonique avec un tenseur diélectrique sous la forme d'un tenseur hermitien non-diagonal. La plus grande partie de ce travail est dédiée à la construction et à l'analyse mathématique d'une telle solution à l'aide du principe d'absorption limite. On montre que cette solution est singulière : elle se décompose en une masse de Dirac à l'origine plus une valeur principale et enfin une fonction régulière de carré sommable. La formule obtenue pour le chauffage du plasma est directement liée à cette singularité.

Keywords: Maxwell equations; Anisotropic dielectric tensor; Hybrid resonance; Resonant heating; Limit absorption principle

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1. Introduction

It is known in plasma physics that Maxwell's equation in the context of a strong background magnetic field may develop singular solutions even for smooth coefficients. This is related to what is called the hybrid resonance [13,20,8] for which we know no mathematical analysis. Hybrid resonance shows up in reflectometry experiments [16,15] and heating devices in fusion plasma [19]. The energy deposit is resonant and may exceed by far the energy exchange which occurs in Landau damping [20,27]. The starting point of the analysis is from the linearization of Vlasov–Maxwell's equations of a non-homogeneous plasma around bulk magnetic field $\mathbf{B}_0 \neq 0$. It yields the non-stationary Maxwell's equations with a linear current

$$\begin{cases} -\dfrac{1}{c^2}\partial_t \mathbf{E} + \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}, \quad \mathbf{J} = -eN_e \mathbf{u}_e, \\ \partial_t \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\ m_e \partial_t \mathbf{u}_e = -e(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0) - m_e \nu \mathbf{u}_e. \end{cases} \tag{1.1}$$

The electric field is $\mathbf{E}$ and the magnetic field is $\mathbf{B}$. The modulus of the background magnetic field $|\mathbf{B}_0|$ and its direction $\mathbf{b}_0 = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}$ will be assumed constant in space for simplicity in our work. The absolute value of the charge of electrons is e, the mass of electrons is m_e, the velocity of light is $c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$ where the permittivity of vacuum is ε_0 and the permeability of vacuum is μ_0. The third equation corresponds to moving electrons with velocity $\mathbf{u}_e$ where the electronic density N_e is a given function of the space variable. One implicitly assumes an ion bath, which is the reason of the friction between the electrons and the ions with collision frequency ν. Much more material about such models can be found in classical physical textbooks [20,8]. The loss of energy in domain Ω can easily be computed in the time domain starting from (1.1). One obtains

$$\frac{d}{dt} \int_{\Omega} \left(\frac{\varepsilon_0 |\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2\mu_0} + \frac{m_e N_e |\mathbf{u}_e|^2}{2} \right) = -\int_{\Omega} \nu m_e N_e |\mathbf{u}_e|^2 + \text{ boundary terms}.$$

Therefore $\mathcal{Q}(\nu) = \int_\Omega \nu m_e N_e |\mathbf{u}_e|^2$ represents the total loss of energy of the electromagnetic field plus the electrons in function of the collision frequency ν. Since the energy loss is necessarily equal to what is gained by the ions, it will be referred to as the heating. We will show that in certain conditions characteristic of the hybrid resonance in frequency domain, the heating does not vanish for vanishing collision friction. So a simple characterization of resonant heating can be written as: $\mathcal{Q}(0^+) > 0$. This apparent paradox is the subject of this work.

As we will prove, the mathematical solution of the time frequency formulation is not square integrable. So that, hybrid resonance is a non-standard phenomenon in the context of the mathematical theory of Maxwell's equations for which we refer to [14,11,26,36]. The situation can be compared with the mathematical theory of metametarials. In [37,38] the electric permittivity and magnetic permeability tensors are degenerate – i.e. they have zero eigenvalues – in surfaces, but they remain positive definite. In this case, the solutions are singular, but the problem remains coercive. See also [12]. In [5,6,4] the coefficient changes in a discontinuous way from being positive to negative. In this situation coerciveness is lost, but as the absolute value of the coefficient is bounded below by a positive constant, the solutions are regular. In our case we have both difficulties at the same time. As the coefficient α (see below) goes from being positive to negative in a continuous way, its absolute value is zero at a point, and, in consequence, our problem is not coercive and there are singular solutions.

1.1. Maxwell's equations in frequency domain

We introduce the notations needed to detail the physics of the problem and to formulate our main result. Writing (1.1) in the frequency domain, that is $\partial_t = -i\omega$, yields

$$\begin{cases} \dfrac{1}{c^2} i\omega \mathbf{E} + \nabla \wedge \mathbf{B} = -\mu_0 e N_e \mathbf{u}_e, \\ -i\omega \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\ -i m_e \omega \mathbf{u}_e = -e(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0) - m_e \nu \mathbf{u}_e. \end{cases} \tag{1.2}$$

One computes the velocity using the third equation