Available online at www.sciencedirect.com
ScienceDirect
J. Math. Pures Appl. 101 (2014) 660–688
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Nash-type equilibria on Riemannian manifolds: A variational approach

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Received 4 March 2013

Available online 9 October 2013

Dedicated to Professor Csaba Varga on the occasion of his 55th birthday

Abstract

Motivated by Nash equilibrium problems on 'curved' strategy sets, the concept of Nash–Stampacchia equilibrium points is introduced via variational inequalities on Riemannian manifolds. Characterizations, existence, and stability of Nash–Stampacchia equilibria are studied when the strategy sets are compact/noncompact geodesic convex subsets of Hadamard manifolds, exploiting two well-known geometrical features of these spaces both involving the metric projection map. These properties actually characterize the non-positivity of the sectional curvature of complete and simply connected Riemannian spaces, delimiting the Hadamard manifolds as the optimal geometrical framework of Nash–Stampacchia equilibrium problems. Our analytical approach exploits various elements from set-valued and variational analysis, dynamical systems, and non-smooth calculus on Riemannian manifolds. Examples are presented on the Poincaré upper-plane model and on the open convex cone of symmetric positive definite matrices endowed with the trace-type Killing form.

Résumé

Motivé par des problèmes d'équilibres de Nash sur des ensembles « courbés » de stratégies, la notion d'équilibre de Nash–Stampacchia peut être introduite par des inégalités variationnelles sur des variétés riemanniennes. On étudie la caractérisation, l'existence et la stabilité d'équilibres de Nash–Stampacchia quand les ensembles de stratégies sont des sous-ensembles géodésiquement convexes, compacts ou non-compacts, de variétés d'Hadamard, en exploitant deux propriétés géométriques bien connues de ces espaces, utilisant la projection métrique. En fait ces propriétés caractérisent la non-positivité de la courbure sectionnelle des espaces de Riemann complets et simplement connexes, en identifiant les variétés d'Hadamard comme structure géométrique optimale oǔ poser les problèmes d'équilibre de Nash–Stampacchia. Notre approche analytique utilise plusieurs éléments d'analyse variationnelle et multi-valuée, des systèmes dynamiques et le calcul non-lisse sur les variétés riemanniennes. Des exemples sont présentés dans le cadre du demi-plan de Poincaré et dans le cône ouvert convexe des matrices définies positives muni d'une forme Killing de type trace.

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[1] Research supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project no. PN-II-ID-PCE-2011-3-0241, "Symmetries in elliptic problems: Euclidean and non-Euclidean techniques", and by János Bolyai Research Scholarship. The present work was initiated during the author's visit at Institut des Hautes Études Scientifiques (IHÉS), Bures-sur-Yvette, France.

0021-7824/$ – see front matter
http://dx.doi.org/10.1016/j.matpur.2013.10.002

Keywords: Nash–Stampacchia equilibrium point; Riemannian manifold; Metric projection; Non-smooth analysis; Non-positive curvature

1. Introduction

After the seminal paper of Nash [30] there has been considerable interest in the theory of Nash equilibria due to its applicability in various real-life phenomena (game theory, price theory, networks, etc.). Appreciating Nash's contributions, R.B. Myerson states that "Nash's theory of non-cooperative games should now be recognized as one of the outstanding intellectual advances of the twentieth century", see also [29]. The Nash equilibrium problem involves n players such that each player know the equilibrium strategies of the partners, but moving away from his/her own strategy alone a player has nothing to gain. Formally, if the sets K_i denote the strategies of the players and $f_i : K_1 \times \cdots \times K_n \to \mathbf{R}$ are their loss-functions, $i \in \{1, \ldots, n\}$, the objective is to find an n-tuple $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K} = K_1 \times \cdots \times K_n$ such that

$$f_i(\mathbf{p}) = f_i(p_i, \mathbf{p}_{-i}) \leqslant f_i(q_i, \mathbf{p}_{-i}) \quad \text{for every } q_i \in K_i \text{ and } i \in \{1, \ldots, n\},$$

where $(q_i, \mathbf{p}_{-i}) = (p_1, \ldots, p_{i-1}, q_i, p_{i+1}, \ldots, p_n) \in \mathbf{K}$. Such point $\mathbf{p}$ is called a *Nash equilibrium point for* $(\mathbf{f}, \mathbf{K}) = (f_1, \ldots, f_n; K_1, \ldots, K_n)$, the set of these points being denoted by $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$.

While most of the known developments in the Nash equilibrium theory deeply exploit the usual convexity of the sets K_i together with the vector space structure of their ambient spaces M_i (i.e., $K_i \subset M_i$), it is nevertheless true that these results are in large part *geometrical* in nature. The main purpose of this paper is to enhance those geometrical and analytical structures which serve as a basis of a systematic study of location of Nash-type equilibria in a general setting as presently possible. In the light of these facts our contribution to the Nash equilibrium theory should be considered intrinsical and analytical rather than game-theoretical. However, it seems that some ideas of the present paper can be efficiently applied to evolutionary game dynamics on curved spaces, see Bomze [5], and Hofbauer and Sigmund [17].

Before to start describing our results, we point out an important (but neglected) topological achievement of Ekeland [15] concerning the *existence* of Nash-type equilibria for two-person games on compact manifolds based on transversality and fixed point arguments. Without the sake of completeness, Ekeland's result says that if $f_1, f_2 : M_1 \times M_2 \to \mathbf{R}$ are continuous functions having also some differentiability properties where M_1 and M_2 are compact manifolds whose Euler–Poincaré characteristics are non-zero (orientable case) or odd (non-orientable case), then there exists at least a Nash-type equilibria for $(f_1, f_2; M_1, M_2)$ formulated via first order conditions involving the terms $\frac{\partial f_i}{\partial x_i}$, $i = 1, 2$.

In the present paper we assume *a priori* that the strategy sets K_i are *geodesic convex* subsets of certain finite-dimensional Riemannian manifolds (M_i, g_i), i.e., for any two points of K_i there exists a unique geodesic in (M_i, g_i) connecting them which belongs entirely to K_i. This approach can be widely applied when the strategy sets are 'curved'. Note that the choice of such Riemannian structures does not influence the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$. As far as we know, the first step into this direction was made recently in [20] via a McClendon-type minimax inequality for acyclic ANRs, guaranteeing the existence of at least one Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$ whenever $K_i \subset M_i$ are compact and geodesic convex sets of certain finite-dimensional Riemannian manifolds (M_i, g_i) while the functions f_i have certain regularity properties, $i \in \{1, \ldots, n\}$. By using Clarke-calculus on manifolds, in [20] we introduced and studied for a wide class of *non*-smooth functions the set of *Nash–Clarke equilibrium points for* $(\mathbf{f}, \mathbf{K})$, denoted in the sequel as $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$; see Section 3. Note that $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ is larger than $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$; thus, a promising way to localize the elements of $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$ is to determine the set $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$ and to choose among these points the appropriate ones. In spite of the naturalness of this approach, we already pointed out its limited applicability due to the involved definition of $\mathcal{S}_{NC}(\mathbf{f}, \mathbf{K})$, conjecturing a more appropriate concept in order to locate the elements of $\mathcal{S}_{NE}(\mathbf{f}, \mathbf{K})$.

Motivated by the latter problem, we observe that the Fréchet and limiting subdifferential calculus of lower semicontinuous functions on Riemannian manifolds developed by Azagra, Ferrera and López-Mesas [1], and Ledyaev and Zhu [23] provides a satisfactory approach. The idea is to consider the following *system of variational inequalities*: find $\mathbf{p} \in \mathbf{K}$ and $\xi_C^i \in \partial_C^i f_i(\mathbf{p})$ such that

$$\left\langle \xi_C^i, \exp_{p_i}^{-1}(q_i) \right\rangle_{g_i} \geqslant 0 \quad \text{for all } q_i \in K_i,\ i \in \{1, \ldots, n\},$$