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J. Math. Pures Appl. 101 (2014) 689–715

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

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The qualitative boundary behavior of blow-up solutions of the super-Liouville equations

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Received 11 July 2013

Available online 4 October 2013

Abstract

Continuing our work on the boundary value problem for super-Liouville equation, we study the qualitative behavior of boundary blow-ups. The boundary condition is derived from the chirality conditions in the physics literature, and is geometrically natural. In technical terms, we derive a new Pohozaev type identity and provide a new alternative, which also works at the boundary, to the classical method of Brézis–Merle.

Résumé

Ce travail est la continuation des nos études du problème des valeurs au bord pour l'équation super-Liouville. Cette équation couple un champ classique de type Liouville avec un champ spinoriel. Cette construction est motivée par la théorie des champs quantiques supersymétriques, et possède une structure variationelle très riche et subtile. On étudie le comportement des solutions de l'équation super-Liouville avec des outils d'analyse géométrique. Plus précisement, on s'intéresse ici au comportement qualitatif des explosions au bord du domaine. Ici, la condition de bord est une version de la condition de chiralité telle qu'elle présentée dans la littérature en physique, mais elle admet aussi une interprétation géométrique trés naturelle. Techniquement, on démontre une identité nouvelle de type Pohozaev, et on présente une approche alternative à la méthode classique du Brézis–Merle, qui s'applique également au bord du domaine.

Keywords: Super-Liouville equations; Chirality and Neumann boundary conditions; Blow-up analysis

1. Introduction

Motivated by the supersymmetric extension of Liouville theory in the recent physics literature, we have constructed a corresponding variational problem that can be studied with the tools of nonlinear analysis (see [12–14]). On one hand, this functional possesses an interesting and rich geometric structure, and on the other hand, the powerful tools of geometric analysis that have been developed since the 1980s allow for a very detailed and precise investigation of the properties of the solutions. In this paper, we carry this program further. In technical terms, we introduce a

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0021-7824/$ – see front matter
http://dx.doi.org/10.1016/j.matpur.2013.10.006

new argument for the blow-up analysis that is based on the removability for a local singularity at the boundary. This argument can not only reprove the classical results on the blow-up behavior of the Liouville equation, but also naturally extends to the boundary situation. In fact, for both physical and geometrical reasons, the boundary behavior is of particular interest. Physically, it corresponds to chirality conditions, and geometrically, it incorporates reflection principles.

Our functional couples the standard Liouville functional with a spinor field term and is therefore called the super-Liouville functional. It is naturally defined on a compact Riemann surface M with or without boundary. The important point is that this generalization preserves a fundamental property of the Liouville functional on Riemann surfaces, namely its conformal invariance. As is well known, conformal invariance is both a key feature in quantum field theory, see the nonlinear sigma model or string theory, and in geometric analysis, see the theory of two-dimensional harmonic maps, minimal surfaces, pseudoholomorphic curves, and the like.

When the domain M is a closed surface, the super-Liouville functional is

$$E(u,\psi)=\int_M\left\{\frac{1}{2}|\nabla u|^2+K_g u+\langle(\not{D}+e^u)\psi,\psi\rangle-e^{2u}\right\}dv,$$

and the Euler–Lagrange system is

$$\begin{cases}-\Delta u=2e^{2u}-e^u\langle\psi,\psi\rangle-K_g, & \text{in } M,\\ \not{D}\psi=-e^u\psi, & \text{in } M.\end{cases} \tag{1}$$

Here M is a Riemann surface with conformal metric g and with a spin structure, K_g is the Gaussian curvature in M, ΣM is the spinor bundle on M with a natural Hermitian product $\langle\cdot,\cdot\rangle$ induced by g, u is a real-valued function on M and ψ is a spinor on M. The Dirac operator $\not{D}$ is defined by $\not{D}\psi:=\sum_{\alpha=1}^2 e_\alpha\cdot\nabla_{e_\alpha}\psi$, where $\{e_1,e_2\}$ is a local orthonormal basis on TM, ∇ is the spin connection on ΣM and $\cdot$ denotes the Clifford multiplication in the spinor bundle ΣM. The Clifford multiplication between e_i and $\psi,\varphi\in\Gamma(\Sigma M)$ satisfies

$$e_i\cdot e_j\cdot\psi+e_j\cdot e_i\cdot\psi=-2\delta_{ij}\psi,\qquad\langle\psi,\varphi\rangle=\langle e_i\cdot\varphi,e_i\cdot\psi\rangle. \tag{2}$$

We refer to [15,10] for more geometric background of spinors and its calculus.

As one knows in geometric analysis, because of conformal invariance, the key for understanding this functional is the blow-up behavior for limits of sequences of solutions. This has been achieved in [12,13].

The purpose of the present paper is to continue the investigation of the more general situation of surfaces with boundary, extending [14]. When the domain M has a nonempty boundary ∂M, the super-Liouville functional becomes

$$E_B(u,\psi)=\int_M\left\{\frac{1}{2}|\nabla u|^2+K_g u+\langle(\not{D}+e^u)\psi,\psi\rangle-e^{2u}\right\}dv+\int_{\partial M}\left(h_g u-ce^u\right)d\sigma,$$

which has been introduced in [14]. Here h_g is the geodesic curvature on ∂M and c is a given constant. In fact, there exists a rich physics literature on this topic, see e.g. [18,2,7].

To continue the discussion about surfaces with boundary, let us first recall the chirality boundary condition (introduced in [8]. See also [9]) for the Dirac operator $\not{D}$, which turns out to be a natural boundary condition for ψ.

We now have to set up the details. Let M be a compact Riemann surface with $\partial M\neq\emptyset$ and with a fixed spin structure, admitting a chirality operator G, which is an endomorphism of the spinor bundle ΣM satisfying:

$$G^2=I,\qquad\langle G\psi,G\varphi\rangle=\langle\psi,\varphi\rangle,\qquad\nabla_X(G\psi)=G\nabla_X\psi,\qquad X\cdot G\psi=-G(X\cdot\psi),$$

for any $X\in\Gamma(TM)$, $\psi,\varphi\in\Gamma(\Sigma M)$. Here I denotes the identity endomorphism of ΣM. Usually, we take $G=\gamma(\omega_2)$, the Clifford multiplication by the complex volume form $\omega_2=ie_1e_2$, where $\{e_1,e_2\}$ is a local orthonormal frame on M.

Denote by $S=\Sigma M|_{\partial M}$ the restricted spinor bundle with induced Hermitian product. Let $\vec{n}$ be the outward unit normal vector field on ∂M. One can verify that $\vec{n}G:\Gamma(S)\to\Gamma(S)$ is a self-adjoint endomorphism satisfying

$$(\vec{n}G)^2=I,\qquad\langle\vec{n}G\psi,\varphi\rangle=\langle\psi,\vec{n}G\varphi\rangle.$$

Hence, we can decompose $S=V^+\oplus V^-$, where $V^\pm$ is the eigensubbundle corresponding to the eigenvalue ±1. One verifies that the orthogonal projection onto the eigensubbundle $V^\pm$: