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Boundary conditions in intrinsic nonlinear elasticity

Philippe G. Ciarlet [a,*], Cristinel Mardare [b]

[a] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
[b] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte Courrier 187, Paris Cedex 05, France*

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Abstract

The intrinsic formulation of the displacement-traction problem of nonlinear elasticity is a system of partial differential equations and boundary conditions whose unknown is the Cauchy–Green strain tensor field instead of the deformation as is customary. We explicitly identify here the boundary conditions satisfied by the Cauchy–Green strain tensor field appearing in such intrinsic formulations.

Résumé

La formulation intrinsèque du problème en déplacement-traction de l'élasticité non linéaire est un système d'équations aux dérivées partielles et de conditions aux limites dont l'inconnue est le champ des tenseurs de déformations de Cauchy–Green, au lieu de la déformation comme il est de coutume. On identifie ici explicitement les conditions aux limites satisfaites par le champ des tenseurs de déformations de Cauchy–Green apparaissant dans de telles formulations intrinsèques.

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Keywords: Intrinsic nonlinear elasticity; Boundary conditions; Cauchy–Green strain tensor

1. Introduction

Throughout this paper, Ω denotes a *bounded*, *connected*, *open* subset of $\mathbb{R}^3$, with a boundary $\Gamma := \partial\Omega$ *of class* $\mathcal{C}^4$, $\boldsymbol{\Phi}_0 : \overline{\Omega} \to \mathbb{R}^3$ denotes an *orientation-preserving immersion of class* $\mathcal{C}^2$, Γ_0 denotes a *non-empty*, *relatively open* subset of Γ, and $\Gamma_1 := \Gamma - \Gamma_0$.

Consider an elastic body with reference configuration $\overline{\Omega}$ and subjected to *applied body and surface forces*, assumed to be *dead loads*, of densities $\boldsymbol{f} : \overline{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{h} : \Gamma_1 \to \mathbb{R}^3$ per unit volume and per unit area, respectively. A *deformation* of the body is a smooth enough immersion $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$ that is *orientation-preserving* and satisfies a *boundary condition of place* imposed on Γ_0. This means that $\det \boldsymbol{\nabla}\boldsymbol{\Phi} > 0$ in $\overline{\Omega}$, where $\boldsymbol{\nabla}\boldsymbol{\Phi}(x)$ denotes at each point $x \in \overline{\Omega}$ the 3×3 gradient matrix of the deformation $\boldsymbol{\Phi}$, and $\boldsymbol{\Phi} = \boldsymbol{\Phi}_0$ on Γ_0.

* Corresponding author.
E-mail address: mapgc@cityu.edu.hk (P.G. Ciarlet).

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The main objective of elasticity theory is to determine the deformation $\boldsymbol{\Phi} : \overline{\Omega} \to \mathbb{R}^3$ undergone by the elastic body in response to these forces and boundary condition. This objective is achieved in two stages (for details about the modeling of three-dimensional nonlinear elasticity see, e.g., Ciarlet [6, Chapters 1–5]). The definitions and notations used, but not defined, here can be found in Section 2.

Thanks to the *stress principle of Euler and Cauchy* and to *Cauchy's theorem*, this amounts to finding the *first Piola–Kirchhoff stress tensor* $\boldsymbol{T} : \overline{\Omega} \to \mathbb{M}^3$ that, together with the deformation $\boldsymbol{\Phi}$, satisfy the following *equations of equilibrium in the reference configuration*:

$$\begin{aligned} -\operatorname{\mathbf{div}} \boldsymbol{T} &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{T}\boldsymbol{n} &= \boldsymbol{h} && \text{on } \Gamma_1, \\ \boldsymbol{\Phi} &= \boldsymbol{\Phi}_0 && \text{on } \Gamma_0, \end{aligned} \tag{1}$$

where $\boldsymbol{n}$ denotes the unit outer normal vector field along Γ_1. Besides, the above equations of equilibrium are supplemented with the *constitutive equation* of an *elastic material*, which takes the following form:

$$\boldsymbol{T}(x) = \widehat{\boldsymbol{T}}\big(x, \nabla\boldsymbol{\Phi}(x)\big) \quad \text{at each } x \in \overline{\Omega}, \tag{2}$$

where $\widehat{\boldsymbol{T}} : \overline{\Omega} \times \mathbb{M}^3_+ \to \mathbb{M}^3$ is a given function, called the *response function* of the elastic material constituting the body. Finally, the *principle of material frame-indifference* implies that there exists a mapping $\widetilde{\boldsymbol{\Sigma}} : \overline{\Omega} \times \mathbb{S}^3_> \to \mathbb{S}^3$, also called a *response function* of the same elastic material, such that

$$\boldsymbol{T}(x) = \nabla\boldsymbol{\Phi}(x)\widetilde{\boldsymbol{\Sigma}}\big(x, \nabla\boldsymbol{\Phi}(x)^T\nabla\boldsymbol{\Phi}(x)\big) \quad \text{at each } x \in \overline{\Omega}. \tag{3}$$

The system formed by Eqs. (1) and (3) constitute the *displacement-traction problem of nonlinear elasticity*.

In the *classical approach to nonlinear elasticity*, the deformation $\boldsymbol{\Phi}$ is considered as the *unknown* of the displacement-traction problem. In the *intrinsic approach to nonlinear elasticity*, the *Cauchy–Green tensor field*

$$\boldsymbol{C} := \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi}$$

is considered as the *unknown* of the displacement-traction problem. This second approach to three-dimensional nonlinear elasticity, which was apparently first suggested (but only briefly) by Antman [2], relies on a classical theorem of differential geometry asserting that a deformation $\boldsymbol{\Phi}$ can be recovered (up to a rigid body motion) from a smooth enough tensor field $\boldsymbol{C}$ if the components of $\boldsymbol{C}$ satisfies specific *compatibility conditions*; cf. Section 4.

Note that an intrinsic approach directly provides, by means of the constitutive equation, the *stress tensor field* $\boldsymbol{T} : \overline{\Omega} \to \mathbb{M}^3$, which is often the unknown of primary interest from the mechanical and computational viewpoints.

The purpose of this paper is to give *explicit expressions of the boundary conditions appearing in the displacement-traction problem of intrinsic nonlinear elasticity*.

First, the boundary condition

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 \quad \text{on } \Gamma_0$$

appearing in the classical formulation of the displacement-traction problem of nonlinear elasticity will be recast in the intrinsic formulation of the same problem as the boundary conditions

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}_0) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}_0) \quad \text{on } \Gamma_0,$$

where $\mathbf{a}^\sharp(\boldsymbol{C})$ and $\mathbf{b}^\sharp(\boldsymbol{C})$ respectively denote the first and second fundamental forms induced on the surface Γ_0 by the Cauchy–Green tensor field $\boldsymbol{C}$, while $\mathbf{a}(\boldsymbol{\varphi}_0)$ and $\mathbf{b}(\boldsymbol{\varphi}_0)$ respectively denote the *first and second fundamental forms induced on Γ_0 by the immersion*

$$\boldsymbol{\varphi}_0 := \boldsymbol{\Phi}_0|_{\Gamma_0}.$$

The definition of these tensor fields (definition (6)) is such that, if $\boldsymbol{C} = \nabla\boldsymbol{\Phi}^T\nabla\boldsymbol{\Phi}$ for some deformation $\boldsymbol{\Phi}$, then

$$\mathbf{a}^\sharp(\boldsymbol{C}) = \mathbf{a}(\boldsymbol{\varphi}) \quad \text{and} \quad \mathbf{b}^\sharp(\boldsymbol{C}) = \mathbf{b}(\boldsymbol{\varphi}) \quad \text{on } \Gamma_0,$$

where $\boldsymbol{\varphi} := \boldsymbol{\Phi}|_{\Gamma_0}$.