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Multiplicity and regularity of solutions for infinitely degenerate elliptic equations with a free perturbation ☆

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ABSTRACT

In this paper, we study the Dirichlet problem for a class of infinitely degenerate elliptic equations with a free perturbation. By using the logarithmic Sobolev inequality, perturbation theorem and Ekeland's variational principle, we obtain the existence of the infinitely many weak solutions and the existence of the nonnegative weak solution. Furthermore, by the methods of iteration and regularity theorems of degenerate elliptic equations, we can also prove the boundedness of the weak solutions and C^∞-regularity of the nonnegative weak solution.

RÉSUMÉ

Dans cet article, on étudie le problème de Dirichlet pour une classe d'équations elliptiques infiniment dégénérées avec des perturbations libres. En utilisant l'inégalité de Sobolev logarithmique, le théorème de perturbation et le principe variationnel d'Ekeland, on obtient l'existence d'une solution faible non negative et aussi infinité de solutions faibles. De plus, par des méthodes d'itération et des théorèmes de régularité des équations elliptiques dégénérées, on montre que les solutions faibles sont bornées et les solutions faibles non negatives ont la régularité C^∞.

1. Introduction and main results

In this paper, we study multiplicity and regularity of the weak solutions for the following infinitely degenerate elliptic equation with a free perturbation,

$$\begin{cases} -\Delta_X u = au\log|u| + bu + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where bounded open domain $\Omega \subset\subset \tilde{\Omega}$, $\tilde{\Omega}$ is an open domain in $\mathbb{R}^n$, a, b are constants, $X = \{X_1, X_2, \cdots, X_m\}$ is C^∞ smooth vector fields defined on $\tilde{\Omega}$, which is infinitely degenerate (i.e. the Hörmander's condition is not satisfied) on a non-characteristic hypersurface $\Gamma \subset \tilde{\Omega}$ and satisfies the finite type of Hörmander's condition with Hörmander index $Q \geq 1$ on $\tilde{\Omega}\backslash\Gamma$. $\Delta_X = \sum_{j=1}^m X_j^2$ is an infinitely degenerate elliptic operator. Here we assume also $\partial\Omega$ is C^∞ smooth and non-characteristic for the system of vector fields X.

$\Delta_X = \sum_{j=1}^m X_j^2$ with finite degeneracy has been studied by Hörmander in [14]. Later, Fefferman–Phong [13] proved that, for a degenerate second order elliptic operator P, sub-elliptic estimate holds iff P is the operator with finite degeneracy (e.g. the Hörmander's condition is satisfied). That means, at degenerate points of infinite type, the sub-elliptic estimate will be not satisfied. However in [15,16], Kohn gave a lot of examples in complex geometry in which the $\bar{\partial}$-Neumann operator is the second order degenerate elliptic operator with infinite type degenerate points. That is, for these infinitely degenerate elliptic operators, the sub-elliptic estimate will be not satisfied. In this case, the hypoellipticity for a class of infinitely degenerate elliptic operators was studied by Morimoto [19] and [20], and later by Christ [12] for general cases. As a replacement of the sub-elliptic estimate, the results in [19] and [12] proved that, if infinitely degenerate elliptic operator $-\Delta_X$ satisfies the following logarithmic regularity estimate:

$$\left\|(\log \Lambda)^s u\right\|^2_{L^2(\Omega)} \leq C_0\left[\int_\Omega |Xu|^2 dx + \|u\|^2_{L^2(\Omega)}\right], \quad \text{for all } u \in C_0^\infty(\tilde{\Omega}), \tag{1.2}$$

where $\Lambda = (e^2 + |D|^2)^{\frac{1}{2}} = \langle D\rangle$ and $s > 1$. Then it is hypo-elliptic.

Related on the vector fields X, let us introduce the following Sobolev space (cf. [23]),

$$H^1_X(\tilde{\Omega}) = \left\{u \in L^2(\tilde{\Omega}) \mid X_j u \in L^2(\tilde{\Omega}),\ j = 1, \cdots, m\right\},$$

which is a Hilbert space with norm $\|u\|^2_{H^1_X} = \|u\|^2_{L^2} + \|Xu\|^2_{L^2}$, and $\|Xu\|^2_{L^2} = \sum_{j=1}^m \|X_j u\|^2_{L^2}$. Let $\Omega \subset\subset \tilde{\Omega}$ be a bounded open subset and suppose that $\partial\Omega$ is C^∞ and non-characteristic for the system of vector fields X. Then the subspace $H^1_{X,0}(\Omega)$ is defined as a closure of $C_0^\infty(\Omega)$ in $H^1_X(\tilde{\Omega})$, which is also a Hilbert space.

In case of $f = 0$, if the infinitely degenerate vector fields X satisfies the logarithmic regularity estimates (1.2), Morimoto–Xu [23] proved the problem (1.1) had a nontrivial solution in $H^1_{X,0}(\Omega)$. Later in [18], for $f = 0$ the authors used the symmetrical mountain pass theorem to prove the existence of infinitely many weak solutions for the problem (1.1). However, in case of $f \neq 0$ in the problem (1.1), there is no symmetry for the energy functional here, the proof of existence of infinitely many weak solutions seems more difficult. In this aspect, we have the following main result.

Theorem 1.1. *If $a > 0$, $f(x) \in L^2(\Omega)$ and X satisfies the logarithmic regularity estimate (1.2) with $s > 1$. Then the problem (1.1) has infinitely many nontrivial weak solutions in $H^1_{X,0}(\Omega)$.*

For the existence of the nonnegative weak solution, one has that

Theorem 1.2. *If X satisfies the logarithmic regularity estimate (1.2) with $s > 1$, $a > 0$; $f(x) \in L^2(\Omega)$ and $f \not\equiv 0$. Then there exists a constant $M > 0$ (see (4.3) below) which is independent of $f(x)$ and depends on the first Dirichlet eigenvalue λ_1 of $-\Delta_X$, such that if $\|f\|_{L^2(\Omega)} < M$, then the problem (1.1) has at least one nontrivial nonnegative weak solution in $H^1_{X,0}(\Omega)$.*

The regularity of the weak solutions can be lifted in the domain of $\Omega\backslash\Gamma$, in fact we have the following regularity result: