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Global existence versus blow-up results for a fourth order parabolic PDE involving the Hessian

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A B S T R A C T

We consider a partial differential equation that arises in the coarse-grained description of epitaxial growth processes. This is a parabolic equation whose evolution is governed by the competition between the determinant of the Hessian matrix of the solution and the biharmonic operator. This model might present a gradient flow structure depending on the boundary conditions. We first extend previous results on the existence of stationary solutions to this model for Dirichlet boundary conditions. For the evolution problem we prove local existence of solutions for arbitrary data and global existence of solutions for small data. By exploiting the boundary conditions and the variational structure of the equation, according to the size of the data we prove finite time blow-up of the solution and/or convergence to a stationary solution for global solutions.

R É S U M É

On considère une équation différentielle qui décrit la croissance épitaxiale d'une couche rugueuse de façon macroscopique. Il s'agit d'une équation parabolique pour laquelle l'évolution est gouvernée par une compétiton entre le déterminant Hessien de la solution et l'opérateur biharmonique. Ce modèle peut présenter une structure de flux gradient suivant les conditions au bord. On étend d'abord des résultats précédents sur l'existence de solutions stationnaires pour ce modèle avec des conditions de Dirichlet. Pour l'équation d'évolution on démontre l'existence locale de solutions pour toute donnée initiale et l'existence globale pour des données suffisamment petites. En exploitant les conditions au bord et la structure variationnelle de l'équation, suivant la taille de la donnée initiale on démontre l'explosion en temps fini et/ou la convergence vers une solution stationnaire pour les solutions globales.

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1. Introduction

Epitaxial growth is a technique by means of which the deposition of new material on existing layers of the same material takes place under high vacuum conditions. It is used in the semiconductor industry for the growth of crystalline structures that might be composed of pure chemical elements like silicon or germanium, or it could instead be formed by alloys like gallium arsenide or indium phosphide. In the case of molecular beam epitaxy the deposition is a very slow process and happens almost atom by atom.

Throughout this paper we assume that $\Omega \subset \mathbb{R}^2$ is an open, bounded smooth domain which is the place where the deposition takes place. Although this kind of mathematical model can be studied in any spatial dimension N, we will concentrate here on the physical situation $N = 2$. The macroscopic evolution of the growth process can be modeled with a partial differential equation that is frequently proposed invoking phenomenological and symmetry arguments [4,31]. The solution of such a differential equation is the function

$$u : \Omega \times \mathbb{R}_+ \to \mathbb{R},$$

describing the height of the growing interface at the spatial location $x \in \Omega$ at the temporal instant $t \in \mathbb{R}_+ := [0, \infty)$. A fundamental modeling assumption in this field is considering that the physical interface can be described as the graph of u, and this is a valid hypothesis in an important number of cases [4].

One of the most widespread examples of this type of theory is the Kardar–Parisi–Zhang equation [25]

$$u_t = \nu \Delta u + \gamma |\nabla u|^2 + \eta(x,t),$$

which has been extensively studied in the physical literature and has also been investigated for its interesting mathematical properties [1,2,5,6,20]. On the other hand, it has been argued that epitaxial growth processes should be described by a different equation coming from a conservation law and, in particular, the term $|\nabla u|^2$ should not be present in such a model [4]. An equation fulfilling these properties is the conservative counterpart of the Kardar–Parisi–Zhang equation [28,38,40]

$$u_t = -\mu \Delta^2 u + \kappa \Delta |\nabla u|^2 + \zeta(x,t). \tag{1}$$

This equation is conservative in the sense that the mean value $\int_\Omega u \, dx$ is constant if boundary conditions that isolate the system are used. It can also be considered as a higher order counterpart of the Kardar–Parisi–Zhang equation. In recent years, much attention has been devoted to other models of epitaxial growth, see [21,26,27,30,41] and references therein.

Herein we will consider a different model obtained by means of the variational formulation developed in [31] and aimed at unifying previous approaches. We skip the detailed derivation of our model, that can be found in [10], and move to the resulting equation, that reads

$$u_t = 2K_1 \det\left(D^2 u\right) - K_2 \Delta^2 u + \xi(x,t).$$

This partial differential equation can be thought of as an analogue of Eq. (1); in fact, they are identical from a strict dimensional analysis viewpoint. Let us also note that this model has been shown to constitute a suitable description of epitaxial growth in the same sense as Eq. (1), and it even displays more intuitive geometric properties [9,12]. The constants K_1 and K_2 will be rescaled in the following.

In this work we are interested in the following initial-boundary value problem:

$$\begin{cases} u_t + \Delta^2 u = \det\left(D^2 u\right) + \lambda f & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \text{boundary conditions} & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2}$$