J. Math. Pures Appl. 103 (2015) 958–1002

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Quantitative homogenization of elliptic partial differential equations with random oscillatory boundary data

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ARTICLE INFO

Article history:
Received 10 August 2014
Available online 13 October 2014

MSC:
35B24
35B40
35R60

Keywords:
Random homogenization
Concentration inequalities
Fully nonlinear elliptic equations
Boundary layers

ABSTRACT

We study the averaging behavior of nonlinear uniformly elliptic partial differential equations with random Dirichlet or Neumann boundary data oscillating on a small scale. Under conditions on the operator, the data and the random media leading to concentration of measure, we prove an almost sure and local uniform homogenization result with a rate of convergence in probability.

RÉSUMÉ

On étudie le comportement homogénéisant d'équations aux dérivées partielles elliptiques nonlinéaires, avec conditions au bord de Dirichlet ou de Neumann aléatoires, oscillantes à petite échelle. Sous certaines contraintes sur l'opérateur, telles que les données et les milieux aléatoires conduisent à une concentration de la mesure, on démontre un résultat presque sûr d'homogénéisation locale uniforme, avec un taux de convergence en probabilité.

1. Introduction

In this article we investigate the averaging behavior of the solutions to nonlinear uniformly elliptic partial differential equations with random Dirichlet or Neumann boundary data oscillating on a small scale. Under conditions on the operator, the data and the random media leading to concentration of measure, we prove an almost sure and local uniform homogenization result with a rate of convergence in probability.

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[1] W.M. Feldman and I.C. Kim both have been partially supported by the NSF grant DMS-1300445.
[2] P.E. Souganidis was partially supported by the NSF grants DMS-0901802 and DMS-1266383.

http://dx.doi.org/10.1016/j.matpur.2014.10.001

In particular, we consider the Dirichlet and Neumann boundary value problems

$$\begin{cases} F(D^2u^\varepsilon) = 0 & \text{in } U, \\ u^\varepsilon = g\left(\cdot, \frac{\cdot}{\varepsilon}, \omega\right) & \text{on } \partial U, \end{cases} \tag{1.1}$$

and

$$\begin{cases} F(D^2u^\varepsilon) = 0 & \text{in } U \setminus K, \\ \partial_\nu u^\varepsilon = g\left(\cdot, \frac{\cdot}{\varepsilon}, \omega\right) & \text{on } \partial U, \\ u^\varepsilon = f & \text{on } \partial K, \end{cases} \tag{1.2}$$

where U is a smooth bounded domain in $\mathbb{R}^d$ with $d \geq 2$, K is a compact subset of U, ν is the inward normal, F is positively homogeneous of degree one and uniformly elliptic, f is continuous on K and $g = g(x, y, \omega)$ is bounded and Lipschitz continuous in x, y uniformly in ω belonging to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and, for each fixed $x \in U$, stationary with respect to the translation action of $\mathbb{R}^d$ on Ω and strongly mixing with respect to (y, ω) (the precise assumptions are given in Section 2.2).

We show that there exist a deterministic continuous functions $\bar{g}_D, \bar{g}_N : \partial U \to \mathbb{R}$ such that, as $\varepsilon \to 0$, the solutions $u^\varepsilon = u^\varepsilon(\cdot, \omega)$ of (1.1) and (1.2) converge almost surely and locally uniformly in U (with a rate in probability) to the unique solution $\bar{u}$ of respectively

$$\begin{cases} F(D^2\bar{u}) = 0 & \text{in } U, \\ \bar{u} = \bar{g}_D & \text{on } \partial U, \end{cases} \tag{1.3}$$

and

$$\begin{cases} F(D^2\bar{u}) = 0 & \text{in } U \setminus K, \\ \partial_\nu \bar{u} = \bar{g}_N & \text{on } \partial U, \\ \bar{u} = f & \text{on } \partial K. \end{cases} \tag{1.4}$$

The homogenized boundary condition $\bar{g}$ (here and in the rest of the paper we omit the subscript and always denote the homogenized boundary condition by $\bar{g}$) depends on F, ν, d and the random field g. The rate of convergence depends on the regularity of U, the continuity and mixing properties of g, the dimension d, the ellipticity ratio of F and, in the case of the Neumann problem, the bounds of f.

We discuss next heuristically what happens as $\varepsilon \to 0$ in the Dirichlet problem (1.1). It follows from the up to the boundary continuity of the solutions to (1.1) that, close to the boundary, u^ε typically has unit size oscillations over distances of order ε. Therefore any convergence to a deterministic limit must be occurring outside of some shrinking boundary layer, where the solution remains random and highly oscillatory even as $\varepsilon \to 0$. In order to analyze the behavior of the u^ε near a point $x_0 \in \partial U$ with inner normal ν we "blow up" u^ε to scale ε, that is we consider

$$v^\varepsilon(y, \omega) = u^\varepsilon(x_0 + \varepsilon y, \omega).$$

If homogenization holds, then $v^\varepsilon(R\nu, \omega)$ should converge to $\bar{g}(x_0)$ for $R > 0$ sufficiently large to escape the boundary layer. Noting that the random function $v^\varepsilon(\cdot, \omega)$ is uniformly continuous, as $\varepsilon \to 0$, we can approximate $v^\varepsilon(\cdot, \omega)$ by the solution of the half-space problem, obtained after "blowing up" in the tangent half-space at x_0,

$$\begin{cases} F(D^2v) = 0 & \text{in } \{y \in \mathbb{R}^d : y \cdot \nu > 0\}, \\ v(\cdot, \omega) = \psi(\cdot, \omega) & \text{on } \{y \in \mathbb{R}^d : y \cdot \nu = 0\}. \end{cases} \tag{1.5}$$