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Stochastic porous media equations in $\mathbb{R}^d$

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ABSTRACT

Existence and uniqueness of solutions to the stochastic porous media equation $dX - \Delta\psi(X)dt = XdW$ in $\mathbb{R}^d$ are studied. Here, W is a Wiener process, ψ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $\psi(r) \le C|r|^m$, $\forall r \in \mathbb{R}$. In this general case, the dimension is restricted to $d \ge 3$, the main reason being the absence of a convenient multiplier result in the space $\mathcal{H} = \{\varphi \in \mathcal{S}'(\mathbb{R}^d);\ |\xi|(\mathcal{F}\varphi)(\xi) \in L^2(\mathbb{R}^d)\}$, for $d \le 2$. When ψ is Lipschitz, the well-posedness, however, holds for all dimensions on the classical Sobolev space $H^{-1}(\mathbb{R}^d)$. If $\psi(r)r \ge \rho|r|^{m+1}$ and $m = \frac{d-2}{d+2}$, we prove the finite time extinction with strictly positive probability.

RÉSUMÉ

One étudie l'existence et l'unicité pour les solutions d'une équation de milieux poreux $dX - \Delta\psi(X)dt = XdW$ dans $\mathbb{R}^d$. Ici W est un processus de Wiener, ψ est un graphe maximal monotone dans $\mathbb{R} \times \mathbb{R}$ tel que $\psi(r) \le C|r|^m$, $\forall r \in \mathbb{R}$. Dans ce contexte général, la dimension est restreinte à $d \ge 3$, essentiellement compte tenu de l'absence d'un résultat adéquat de multiplication dans l'espace $\mathcal{H} = \{\varphi \in \mathcal{S}'(\mathbb{R}^d);\ |\xi|(\mathcal{F}\varphi)(\xi) \in L^2(\mathbb{R}^d)\}$, pour $d \le 2$. Lorsque ψ est Lipschitz, le problème est néanmoins bien posé pour toute dimension dans l'espace de Sobolev classique $H^{-1}(\mathbb{R}^d)$. Si $\psi(r)r \ge \rho|r|^{m+1}$ et $m = \frac{d-2}{d+2}$, on démontre une propriété d'extinction en temps fini avec probabilité strictement positive.

1. Introduction

Consider the stochastic porous media equation

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$$dX - \Delta\psi(X)dt = XdW \quad \text{in } (0,T)\times\mathbb{R}^d,$$
$$X(0) = x \quad \text{on } \mathbb{R}^d, \tag{1.1}$$

where ψ is a monotonically nondecreasing function on $\mathbb{R}$ (eventually multivalued) and $W(t)$ is a Wiener process of the form

$$W(t) = \sum_{k=1}^{\infty} \mu_k e_k \beta_k(t), \quad t \geq 0. \tag{1.2}$$

Here $\{\beta_k\}_{k=1}^{\infty}$ are independent Brownian motions on a stochastic basis $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, $\mu_k \in \mathbb{R}$ and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis in $H^{-1}(\mathbb{R}^d)$ or $\mathcal{H}^{-1}$ (see (2.2) below) to be made precise later on.

On bounded domains $\mathcal{O} \subset \mathbb{R}^d$ with Dirichlet homogeneous boundary conditions, Eq. (1.1) was studied in [3–5], under general assumptions on $\psi : \mathbb{R} \to \overline{\overline{\mathbb{R}}}$ (namely, maximal monotone multivalued graph with polynomial growth, or even more general growth conditions in [4]). It should be said, however, that there is a principal difference between bounded and unbounded domains, mainly due to the multiplier problem in Sobolev spaces on $\mathbb{R}^d$. If $d \geq 3$ and $\mathcal{O} = \mathbb{R}^d$, existence and uniqueness of solutions to (1.1) were proved in [21] (see, also, [23]) in a general setting which covers the case $\mathcal{O} = \mathbb{R}^d$ (see Theorem 3.9, Proposition 3.1 and Example 3.4 in [21]). However, it should be said that in [21] ψ is assumed continuous, such that $r\psi(r) \to \infty$ as $r \to \infty$, which we do not need in this paper.

We study the existence and uniqueness of (1.1) under two different sets of conditions requiring a different functional approach. The first one, which will be presented in Section 3, assumes that ψ is monotonically nondecreasing and Lipschitz. The state space for (1.1) is, in this case, $H^{-1}(\mathbb{R}^d)$, that is, the dual of the classical Sobolev space $H^1(\mathbb{R}^d)$. In spite of the apparent lack of generality (ψ Lipschitz), it should be mentioned that there are physical models described by such an equation as, for instance, the two phase Stefan transition problem perturbed by a stochastic Gaussian noise [2]; moreover, in this latter case there is no restriction on the dimension d.

The second case, which will be studied in Section 4, is that where ψ is a maximal monotone multivalued function with at most polynomial growth. An important physical problem covered by this case is the self-organized criticality model

$$dX - \Delta H(X - X_c)dt = (X - X_c)dW, \tag{1.3}$$

where H is the Heaviside function and X_c is the critical state (see [5,6,8]). More generally, this equation with discontinuous ψ covers the stochastic nonlinear diffusion equation with singular diffusivity $D(u) = \psi'(u)$.

It should be mentioned that, in this second case, the solution $X(t)$ to (1.1) is defined in a certain distribution space $\mathcal{H}^{-1}$ (see (2.2) below) on $\mathbb{R}^d$ and the existence is obtained for $d \geq 3$ only, as in the case of continuous ψ in [21]. The case $1 \leq d \leq 2$ remains open due to the absence of a multiplier rule in the norm $\|\cdot\|_{\mathcal{H}^{-1}}$ (see Lemma 4.1 below).

In Section 5, we prove the finite time extinction of the solution X to (1.1) with strictly positive probability under the assumption that $\psi(r)r \geq \rho|r|^{m+1}$ and $m = \frac{d-2}{d+2}$.

Finally, we would like to comment on one type of noise. Existence and uniqueness can be proved with $g(t, X(t))$ by replacing $X(t)$ under (more or less the usual) abstract conditions on σ (see, e.g., [21,23]). The main reason why in this paper we restrict ourselves to linear multiplicative noise is that first we want to be concrete, second the latter case is somehow generic (just think of taking the Taylor expansion of $\sigma(t,\cdot)$ up to first order), and third for this type of noise we prove finite time extinction in Section 5.