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The Heston Riemannian distance function

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Abstract

The Heston model is a popular stock price model with stochastic volatility that has found numerous applications in practice. In the present paper, we study the Riemannian distance function associated with the Heston model and obtain explicit formulas for this function using geometrical and analytical methods. Geometrical approach is based on the study of the Heston geodesics, while the analytical approach exploits the links between the Heston distance function and the Carnot–Carathéodory distance function in the Grushin plane. For the Grushin plane, we establish an explicit formula for the Legendre–Fenchel transform of the limiting cumulant generating function and prove a partial large deviation principle that is true only inside a special set.

Résumé

Le modèle de Heston est un modèle standard pour un actif gouverné par une volatilité stochastique. Ce modèle a trouvé de nombreuses applications dans la pratique. Dans cet article on étudie la distance riemanniene associée au modèle de Heston et on obtient des formules explicites pour cette distance en utlisant des méthodes géométriques ainsi que des méthodes analytiques. L'approche géométrique utilise les géodésiques dans le modèle de Heston, alors que l'approche analytique utilise le lien qui existe entre la distance de Carnot–Carathéodory dans le plan de Grushin et la distance de Heston. Dans le plan de Grushin, on établit une formule explicite pour la transformée de Legendre–Fenchel de la fonction génératrice limite cumulante et on démontre un principe de grandes déviations partielles qui est valable dans un ensemble qu'on identifie.

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1. Introduction

There are two main protagonists in this paper: the Riemannian manifold associated with the Heston model of the stock price, and the Grushin plane, which is one of the best-known examples of a Carnot–Carathéodory space. The present paper focuses on the Heston Riemannian distance and the Carnot–Carathéodory distance in the Grushin plane

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and provides explicit formulas for these. The Heston distance and the Grushin distance are intimately related, and various facts concerning these distances can be easily transplanted from one setting into the other.

We will next briefly describe the main results obtained in this paper. Theorems 2 and 6 below contain explicit formulas for the Heston distance. The formulas in Theorem 2 are established using geometrical approach, hereafter referred to as *C-approach*, while the proof of the distance formula in Theorem 6 uses certain links between the Heston and the Grushin distances and is more analytical. The second approach will be referred to as *δ-approach*. In the proof of Theorem 6, we compute and study the limiting cumulant generating function Λ for the Grushin plane and the Legendre–Fenchel transform Λ^* of the function Λ. One of the main results in the present paper is a partial large deviation principle for the Grushin plane (see Theorem 24). The word "partial" is used in the previous sentence because in the case of the Grushin plane the large deviation principle with Λ^* as a rate function holds only inside a special subset of $\mathbb{R}^2$.

We would also like to bring to the reader's attention certain key qualitative properties, present in both the δ- and in the C-approach, which greatly facilitate the efficient and rapid numerical determination of the joint Heston distance function. In both approaches, this determination is reduced to the solution of a *single* transcendental equation in one variable, which is proved to be convex in the δ-approach and convex or monotonic in the C-approach (see Lemmas 10, 11, and 12). These lemmas ensure that the equations can be efficiently and rapidly solved by Newton's method or a bisection method. We also show in the present paper that it is crucial to distinguish two different regimes (the near and the far point regime) in the geometrical and analytical approaches to the Heston distance, each regime requiring it's own analysis (see Theorems 2 and 6 and their proofs).

Let us expand on the financial motivations for considering the Heston Riemannian distance function. The Heston model is one of the most popular stock price models with stochastic volatility. This model was introduced in [22]. The stock price process S and the variance process V in the Heston model satisfy the following system of stochastic differential equations:

$$\begin{cases} dS_t = \mu S_t\,dt + \sqrt{V_t}S_t\,dW_t, \\ dV_t = (a - bV_t)\,dt + c\sqrt{V_t}\,dZ_t, \end{cases} \tag{1}$$

where $a \geqslant 0$, $b \geqslant 0$, $c > 0$. In (1), W and Z are correlated standard Brownian motions such that $d\langle W, Z\rangle_t = \rho\,dt$ with $\rho \in (-1, 1)$. We refer the reader to [17,18,23,35] for more information on the Heston model and stochastic volatility models.

Riemannian geometry has found numerous applications in mathematical finance (a relevant reference here is the book [23] by P. Henry-Labordère). For example, a key element in determining the term structure of the implied volatility in the Heston model is the Riemannian distance d to a line in the Heston half-plane (see [2,13,14,20,23,30]). Another important characteristic of various stochastic models is the Legendre–Fenchel transform Λ^* of the limiting cumulant generating function. This function is intimately related to the Riemannian distance function. More precisely, under certain restrictions, the equality $\frac{d^2}{2} = \Lambda^*$ holds, and moreover, the function Λ^* plays the role of the rate function in the large deviation principle on the Riemannian manifold associated with the model. For the log-price process in the Heston model, such results were obtained in the paper [13] of M. Forde and A. Jacquier (see also [15,16]). On the other hand, for the two-dimensional Grushin model associated with the log-price and the variance process in the Heston model, the large deviation principle and the equality $\frac{d^2}{2} = \Lambda^*$ hold only in a part of the plane $\mathbb{R}^2$ (Theorems 24 and 25, respectively). The previous facts contrast sharply with the corresponding results in [13,16].

Let us first consider the following uncorrelated Heston model:

$$\begin{cases} dS_t = S_t\sqrt{V_t}\,dW_t, \\ dV_t = (a - bV_t)\,dt + \sqrt{V_t}\,dZ_t, \end{cases} \tag{2}$$

where $a \geqslant 0$, $b \geqslant 0$, and W and Z are independent standard Brownian motions. Denote by X the log-price process defined by $X = \log S$. Then the model in (2) transforms as follows:

$$\begin{cases} dX_t = -\frac{1}{2}V_t\,dt + \sqrt{V_t}\,dW_t, \\ dV_t = (a - bV_t)\,dt + \sqrt{V_t}\,dZ_t. \end{cases} \tag{3}$$

The state space for the process (X, V) is the closed half-plane