J. Math. Pures Appl. 103 (2015) 68–101

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

First-order expansion of homogenized coefficients under Bernoulli perturbations

Jean-Christophe Mourrat

EPFL, Institut de mathématiques, station 8, 1015 Lausanne, Switzerland

ARTICLE INFO

Article history:
Received 4 April 2013
Available online 20 March 2014

MSC:
35B27
35J15
35R60
82D30

Keywords:
Homogenization
Random media
Clausius–Mossotti formula
Bernoulli perturbation

ABSTRACT

Divergence-form operators with stationary random coefficients homogenize over large scales. We investigate the effect of certain perturbations of the medium on the homogenized coefficients. The perturbations considered are rare at the local level, but when occurring, have an effect of the same order of magnitude as the initial medium itself. The main result of the paper is a first-order expansion of the homogenized coefficients, as a function of the perturbation parameter.

RÉSUMÉ

Les opérateurs sous forme divergence à coefficients aléatoires stationnaires s'homogénéisent sur les grandes échelles spatiales. On s'interesse à l'effet de certaines perturbations aléatoires du milieu sur les coefficients homogénéisés. Les perturbations considérées sont rares au niveau local, mais lorsqu'elles ont lieu, elles ont un effet du même ordre de grandeur que le milieu initial lui-même. Le résultat principal de l'article est un développement limité au premier ordre des coefficients homogénéisés, en fonction du paramètre de perturbation.

1. Introduction

Consider a divergence-form operator whose coefficients are random. The randomness of the coefficients model the small-scale irregularity of a medium. If the distribution of the coefficients is stationary and ergodic (and if some ellipticity condition holds), then this random operator can, over large scales, be replaced by an effective operator with constant *homogenized* coefficients.

The aim of this paper is to study the effect of certain perturbations of the medium on the homogenized coefficients. The perturbations considered are small only in the sense that locally, the medium is perturbed with small probability; but where a perturbation occurs, the change is of the same order of magnitude as the medium itself. This type of perturbation may be called a *Bernoulli perturbation*. Our main purpose is to

http://dx.doi.org/10.1016/j.matpur.2014.03.008
0021-7824/

prove a first-order expansion of the homogenized coefficients, as a function of the perturbation parameter, under conditions of short-range correlations and uniform ellipticity of the medium.

There are at least two important motivations behind this problem. The first concerns the numerical approximation of the homogenized coefficients. Although several techniques for doing so have been identified and analysed [18], it remains a computationally expensive task, even in low dimensions. It has thus been proposed in [2] to study more efficient techniques that would apply to *weakly random* media, that is, random perturbations of a periodic environment. First-order expansions (as a function of the perturbation parameter) have been proved in [3], but only for specific types of perturbations that do not include Bernoulli perturbations. Yet, Bernoulli perturbations are arguably the most natural modelling assumption for typical disordered media like composite materials (as an example, see [4] for a cross-section of a composite material used in the aeronautics industry). In [4,3], conjectures are formulated concerning the expansion of the homogenized coefficients for such perturbations, which are backed by a formal derivation and numerical evidence.

A second motivation is related to percolation. For this model, each edge of $\mathbb{Z}^d$ is independently removed with probability p, and kept otherwise. There exists a critical probability $p_c \in (0, 1)$ such that the remaining graph has a unique infinite connected component if $p < p_c$, and has only finite connected components if $p > p_c$. Over the last years, the understanding of two-dimensional percolation close to criticality progressed tremendously, in particular through the rigorous derivation of the values of several critical exponents (see for instance [37]). Yet, to the best of my knowledge, it is not known (for any $d \geqslant 2$) how the homogenized conductivity of the percolation cluster behaves as p approaches p_c from below (see [22] for a nice review of the problem). Understanding the effect of small perturbations of the medium on the homogenized conductivity seems to be a first necessary step towards the resolution of this problem.

Of the two motivations above, the first is formulated for differential operators, while the second is inherently discrete. In what follows, we focus on a discrete model. Apart from this difference, the results presented here give a proof of the first-order expansion of the homogenized coefficients conjectured in [2,4,3]. It extends it in the sense that the non-perturbed environment considered here need not be periodic, and that the expansion is obtained around every value of the perturbation parameter. We work under a condition of uniform ellipticity, which is obviously not satisfied in the case of percolation. The present paper will hopefully lay the basis for an extension to the case of percolation, as well as to higher-order expansions.

On a heuristic level, the first-order expansion of the homogenized coefficients can be guessed as follows. The homogenized coefficients can be expressed in terms of the *corrector*, whose defining equation is posed on the whole space. It is well-known that finite-volume approximations of the corrector yield consistent approximations as the volume tends to infinity. One can easily derive a first-order expansion of the corrector defined over a given finite volume element (or any reasonable function of it) as the perturbation parameter tends to 0. Indeed, in this limit, one can assume at first order that there is at most one location that is perturbed. One then gets a formula for the first-order expansion of the homogenized coefficients by formally interchanging the "infinite-volume" and the "small perturbation parameter" limits.

We will see here that this informal reasoning can be made rigorous, and thus provide us with a first-order expansion of the homogenized coefficients. The main point is to quantify errors when localizing the problem over a finite volume, say of side length N, and then choose N as a function of the perturbation parameter p. It is clear that, for this strategy to make sense at all, we need the box to contain some perturbed locations, so we should have $N^d \gg p^{-1}$. A more refined heuristic consists in observing that a random walk evolving in a box of size N sees only of order N^2 sites, so that we should in fact need $N^2 \gg p^{-1}$. Using parabolic/elliptic regularity theory, the averaged estimates on the gradients of the Green function of [17], and the localization error estimates on the corrector due to [20,21], we will see that the argument can be made rigorous if we choose $N^2 = p^{-(1+\varepsilon)}$ for some small $\varepsilon > 0$. (One should think of N as being $\mu^{-1/2}$ in the notation introduced below.)