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Compactness of trajectories to some nonlinear second order evolution equations and applications

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Abstract

Under suitable growth and coercivity conditions on the nonlinear damping operator g, we establish boundedness or compactness properties of trajectories to the equation

$$\ddot{u}(t) + g\big(\dot{u}(t)\big) + Au(t) = h(t), \quad t \in \mathbb{R}_+,$$

where A is a positive selfadjoint operator. The compactness results are used to prove the existence of almost periodic solutions when h is almost periodic, and to generalize some recent results of Chergui and Ben Hassen–Chergui concerning convergence to equilibrium when a nonlinear term depending on u is added and h dies off sufficiently fast for t large.

Résumé

Sous des conditions de croissance et de coercivité sur l'opérateur de dissipation g, on établit des propriétés de bornage et de compacité des trajectoires de l'équation

$$\ddot{u}(t) + g\big(\dot{u}(t)\big) + Au(t) = h(t), \quad t \in \mathbb{R}_+,$$

où A est un opérateur auto-adjoint positif. Les résultats de compacités sont utilisés pour construire des solutions presque périodiques si h est presque périodique, et pour généraliser des resultats de Chergui et Ben Hassen–Chergui sur la convergence vers l'équilibre en présence d'une non-linéarité portant sur u, lorsque h tend vers 0 assez vite pour t grand.

MSC: 34A34; 34D20; 35B40; 35L10; 35L90

Keywords: Second order equation; Compactness; Almost periodic; Łojasiewicz inequality

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1. Introduction

In this paper we investigate the asymptotic behavior of solutions to the equation

$$\ddot{u}(t) + g\big(\dot{u}(t)\big) + Au(t) + f\big(u(t)\big) = h(t), \quad t \in \mathbb{R}_+, \tag{1.1}$$

where V is a real Hilbert space, $A \in L(V, V')$ is a symmetric, positive, coercive operator, $g \in C(V, V')$ is monotone, f is a gradient operator satisfying some appropriate conditions and h is a forcing term. We are especially interested in the two following cases:

1) $f = 0$ and h is almost periodic.

2) h tends to 0 sufficiently fast at infinity in t and f is the gradient of an analytic functional or more generally the gradient of a potential satisfying the Łojasiewicz gradient inequality in a sense which will be specified later.

Case 1) has been intensively studied in the Literature, covering the following topics: existence of almost periodic solutions, asymptotic behavior of the general solution, rate of decay to 0 of the difference of two solutions in the energy space in the best cases, cf. e.g. [1,5,6,13–15,26,20,27]. Until now, although boundedness of trajectories for $t \geqslant 0$ is sufficient in the linear case and when h is time-periodic, the treatment of the general case has always required the existence of a precompact trajectory, and the problem is that it is difficult to distinguish between different trajectories at this level: if we were able to exhibit a precompact orbit without knowing anything about the others, it would mean that we can (by going to the positive ω-limit set) localize an almost periodic orbit and this is precisely what becomes impossible in the nonlinear case. Therefore we have to prove compactness of all trajectories or nothing (note that for g tangent to 0 at the origin, extra regularity of the initial state or even of the forcing term will not help when t becomes large). Another difficulty is the following: if the existence of bounded trajectories can be proved by combining a coerciveness property of g "at infinity" with some growth restrictions, compactness requires a global, uniform kind of coerciveness. In the past more and more general compactness results have been obtained [15], but only when g is a Nemytskii type operator. When g is a non-local operator or involves differential operators in space, the theory remained to be done: this is the main object of the present paper. At this point we mention the more recent general results of Couchouron [12] on quasi-autonomous dissipative systems relying precompactness of trajectories to uniform continuity on the half-line with values in the phase space. These results are essentially limited to the case where the forcing term is the sum of an integrable function on the half-line and a function with bounded variation. In particular they cannot be used when the forcing term is a non-trivial almost periodic function.

For case 2), compactness is the vital starting point. In the past several significant advances have been done in this direction, cf. e.g. [23,17–19,21,11,9,10,2–4], the other tool here being the Łojasiewicz gradient inequality [24,25]. But here even the case $h = 0$ is non-trivial since the set of equilibria needs not have any particular structure except for the restrictions induced by the existence of a Łojasiewicz inequality: we know for instance in advance that the potential energy is constant on continua inside the set of equilibria, a property which can fail for C^∞ and even Gevrey potentials. The fact that precompactness of trajectories had been proved only for Nemytskii type damping operators limited until now the convergence results with nonlinear damping to those damping operators. Therefore the second innovation of this paper is to contain the first convergence results in case 2) in presence of a non-local damping term.

The plan of the paper is as follows: In Section 2, we introduce the basic tools used in the statements and proofs of the main results. Section 3 is devoted to the initial value problem for (1.1). Sections 4 and 5 contain the statement and proof of the boundedness result and the compactness result, respectively. Sections 6 and 7 contain respectively the statement and proof of the asymptotic almost periodicity in case 1) and the convergence result in case 2), respectively. Finally Section 8 is devoted to the application to PDE models with non-local damping terms.

2. Some useful tools

In this section, we collect a few results of general interest which will turn out essential for the proofs of our main results. We also need to recall the definitions of some well-known mathematical objects as well as their basic properties in the exact functional framework that shall be used in the main sections containing our new results.