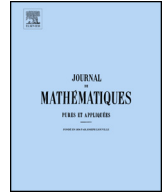




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## A spectral result for Hardy inequalities

Baptiste Devyver

Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

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### ABSTRACT

Let  $P$  be a linear, elliptic second order symmetric operator, with an associated quadratic form  $q$ , and let  $W$  be a non-negative potential such that the Hardy inequality

$$\lambda_0 \int_{\Omega} W u^2 \leq q(u)$$

holds with positive best constant  $\lambda_0$ . We give sufficient conditions so that the spectrum of the operator  $\frac{1}{W}P$  is  $[\lambda_0, \infty)$ . In particular, we apply this to several well-known Hardy inequalities: (improved) Hardy inequalities on a bounded convex domain of  $\mathbb{R}^n$  with potentials involving the distance to the boundary, and Hardy inequalities for minimal submanifolds of  $\mathbb{R}^n$ .

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### R É S U M É

Soit  $P$  un opérateur linéaire elliptique du second ordre,  $q$  sa forme quadratique associée, et  $W$  un potentiel positif tel que l'inégalité de Hardy

$$\lambda_0 \int_{\Omega} W u^2 \leq q(u)$$

soit vérifiée avec la meilleure constante  $\lambda_0$ . On donne des conditions suffisantes pour le spectre de l'opérateur  $\frac{1}{W}P$  soit  $[\lambda_0, \infty)$ . En particulier, on applique ceci à plusieurs inégalités de Hardy bien connues : inégalités de Hardy améliorées sur un domaine convexe borné de  $\mathbb{R}^n$  avec un potentiel dépendant de la distance au bord, et inégalités de Hardy pour des sous-variétés minimales de  $\mathbb{R}^n$ .

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E-mail addresses: [baptiste.devyver@univ-nantes.fr](mailto:baptiste.devyver@univ-nantes.fr), [devyver@tx.technion.ac.il](mailto:devyver@tx.technion.ac.il).

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## 1. Introduction

Let  $P$  be a linear elliptic, second order, symmetric, non-negative operator on a domain  $\Omega$ , and let  $q$  be the quadratic form associated to  $P$ . Following Carron [12] and Tertikas [30], we will call *Hardy inequality* for  $P$  with weight  $W \geq 0$  and constant  $\lambda > 0$ , the following inequality:

$$\lambda \int_{\Omega} W u^2 \leq q(u), \quad \forall u \in C_0^\infty(\Omega). \quad (1.1)$$

We denote by  $\lambda_0 = \lambda_0(\Omega, P, W)$  the best constant  $\lambda$  for which inequality (1.1) is valid. By convention, if (1.1) does not hold for any  $\lambda > 0$ , we will let  $\lambda_0 = 0$ . The inequality (1.1) aims to quantify the positivity of  $P$ : for instance, inequality (1.1) with  $W \equiv 1$  is equivalent to the positivity of the bottom of the spectrum of (the Friedrichs extension of)  $P$ .

Let us give a celebrated example of Hardy inequality for  $P = -\Delta$ , which will be a guideline for us in this paper (see [20] for the convex case, and [7,22] for the mean convex case):

**Example 1.1.** If  $\Omega$  is a  $C^2$ , bounded, mean convex domain of  $\mathbb{R}^n$ , and  $\delta$  is the distance to the boundary of  $\Omega$ , then

$$\frac{1}{4} \int_{\Omega} \frac{u^2}{\delta^2} \leq \int_{\Omega} |\nabla u|^2, \quad \forall u \in C_0^\infty(\Omega). \quad (1.2)$$

We recall that  $\Omega$  is called *mean convex* if the mean curvature of its boundary is non-negative. Let us return to the general case.

### 1.1. The best constant and the existence of minimizers

A natural question is, for a given weight  $W \geq 0$  such that the Hardy inequality (1.1) holds, to compute the best constant  $\lambda_0$  and to discuss whether  $\lambda_0$  is attained by a minimizer in the appropriate space or not. More precisely, define  $\mathcal{D}^{1,2}$  to be the completion of  $C_0^\infty(\Omega)$  with respect to the norm  $\sqrt{q}$ . The variational problem associated to the Hardy inequality (1.1) is

$$\lambda_0 = \inf_{u \in \mathcal{D}^{1,2} \setminus \{0\}} \frac{q(u)}{\int_{\Omega} W u^2}. \quad (1.3)$$

If (1.3) is not realized by a function in  $\mathcal{D}^{1,2}$ , we will say that the Hardy inequality (1.1) with best constant  $\lambda_0$  does not have a minimizer.

Another interesting quantity, related to the existence of minimizers, is the *best constant at infinity*. It is defined as follows (see [2,20,15]):

**Definition 1.2.** The *best constant at infinity*  $\lambda_\infty = \lambda_\infty(\Omega, P, W)$  is the supremum of the set of  $\alpha \geq 0$  such that

$$\alpha \int_{\Omega \setminus K_\alpha} W u^2 \leq q(u), \quad \forall u \in C_0^\infty(\Omega \setminus K_\alpha),$$

for some  $K_\alpha \subset\subset \Omega$  compact subset of  $\Omega$ .

Closely related notions have been introduced (under different names) in [30] and [17].

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