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A spectral result for Hardy inequalities

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ABSTRACT

Let P be a linear, elliptic second order symmetric operator, with an associated quadratic form q, and let W be a non-negative potential such that the Hardy inequality

$$\lambda_0 \int_\Omega W u^2 \leqslant q(u)$$

holds with positive best constant λ_0. We give sufficient conditions so that the spectrum of the operator $\frac{1}{W}P$ is $[\lambda_0, \infty)$. In particular, we apply this to several well-known Hardy inequalities: (improved) Hardy inequalities on a bounded convex domain of $\mathbb{R}^n$ with potentials involving the distance to the boundary, and Hardy inequalities for minimal submanifolds of $\mathbb{R}^n$.

RÉSUMÉ

Soit P un opérateur linéaire elliptique du second ordre, q sa forme quadratique associée, et W un potentiel positif tel que l'inégalité de Hardy

$$\lambda_0 \int_\Omega W u^2 \leqslant q(u)$$

soit vérifiée avec la meilleure constante λ_0. On donne des conditions suffisantes pour le spectre de l'opérateur $\frac{1}{W}P$ soit $[\lambda_0, \infty)$. En particulier, on applique ceci à plusieurs inégalités de Hardy bien connues : inégalités de Hardy améliorées sur un domaine convexe borné de $\mathbb{R}^n$ avec un potentiel dépendant de la distance au bord, et inégalités de Hardy pour des sous-variétés minimales de $\mathbb{R}^n$.

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1. Introduction

Let P be a linear elliptic, second order, symmetric, non-negative operator on a domain Ω, and let q be the quadratic form associated to P. Following Carron [12] and Tertikas [30], we will call *Hardy inequality* for P with weight $W \geqslant 0$ and constant $\lambda > 0$, the following inequality:

$$\lambda \int_\Omega W u^2 \leqslant q(u), \quad \forall u \in C_0^\infty(\Omega). \tag{1.1}$$

We denote by $\lambda_0 = \lambda_0(\Omega, P, W)$ the best constant λ for which inequality (1.1) is valid. By convention, if (1.1) does not hold for any $\lambda > 0$, we will let $\lambda_0 = 0$. The inequality (1.1) aims to quantify the positivity of P: for instance, inequality (1.1) with $W \equiv 1$ is equivalent to the positivity of the bottom of the spectrum of (the Friedrichs extension of) P.

Let us give a celebrated example of Hardy inequality for $P = -\Delta$, which will be a guideline for us in this paper (see [20] for the convex case, and [7,22] for the mean convex case):

Example 1.1. If Ω is a C^2, bounded, mean convex domain of $\mathbb{R}^n$, and δ is the distance to the boundary of Ω, then

$$\frac{1}{4} \int_\Omega \frac{u^2}{\delta^2} \leqslant \int_\Omega |\nabla u|^2, \quad \forall u \in C_0^\infty(\Omega). \tag{1.2}$$

We recall that Ω is called *mean convex* if the mean curvature of its boundary is non-negative.

Let us return to the general case.

1.1. The best constant and the existence of minimizers

A natural question is, for a given weight $W \geqslant 0$ such that the Hardy inequality (1.1) holds, to compute the best constant λ_0 and to discuss whether λ_0 is attained by a minimizer in the appropriate space or not. More precisely, define $\mathcal{D}^{1,2}$ to be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\sqrt{q}$. The variational problem associated to the Hardy inequality (1.1) is

$$\lambda_0 = \inf_{u \in \mathcal{D}^{1,2} \setminus \{0\}} \frac{q(u)}{\int_\Omega W u^2}. \tag{1.3}$$

If (1.3) is not realized by a function in $\mathcal{D}^{1,2}$, we will say that the Hardy inequality (1.1) with best constant λ_0 does not have a minimizer.

Another interesting quantity, related to the existence of minimizers, is the *best constant at infinity*. It is defined as follows (see [2,20,15]):

Definition 1.2. The *best constant at infinity* $\lambda_\infty = \lambda_\infty(\Omega, P, W)$ is the supremum of the set of $\alpha \geqslant 0$ such that

$$\alpha \int_{\Omega \setminus K_\alpha} W u^2 \leqslant q(u), \quad \forall u \in C_0^\infty(\Omega \setminus K_\alpha),$$

for some $K_\alpha \subset\subset \Omega$ compact subset of Ω.

Closely related notions have been introduced (under different names) in [30] and [17].