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Regularity of $\mathcal{C}^1$ and Lipschitz domains in terms of the Beurling transform

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Abstract

Let $\Omega \subset \mathbb{C}$ be a bounded $\mathcal{C}^1$ domain, or a Lipschitz domain "flat enough", and consider the Beurling transform of χ_Ω:

$$B\chi_\Omega(z) = \lim_{\varepsilon\to 0} \frac{-1}{\pi} \int_{w\in\Omega,\ |z-w|>\varepsilon} \frac{1}{(z-w)^2}\,dm(w).$$

Using a priori estimates, in this paper we solve the following free boundary problem: if $B\chi_\Omega$ belongs to the Sobolev space $W^{\alpha,p}(\Omega)$ for $0<\alpha\leqslant 1$, $1<p<\infty$ such that $\alpha p>1$, then the outward unit normal N on $\partial\Omega$ is in the Besov space $B^{\alpha-1/p}_{p,p}(\partial\Omega)$. The converse statement, proved previously by Cruz and Tolsa, also holds. So we have

$$B(\chi_\Omega) \in W^{\alpha,p}(\Omega) \quad\Longleftrightarrow\quad N \in B^{\alpha-1/p}_{p,p}(\partial\Omega).$$

Together with recent results by Cruz, Mateu and Orobitg, from the preceding equivalence one infers that the Beurling transform is bounded in $W^{\alpha,p}(\Omega)$ if and only if the outward unit normal N belongs to $B^{\alpha-1/p}_{p,p}(\partial\Omega)$, assuming that $\alpha p>2$.

Résumé

Soit $\Omega \subset \mathbb{C}$ un domaine $\mathcal{C}^1$ borné ou un domaine Lipschitz assez plat. On considère la transformée de Beurling de χ_Ω :

$$B\chi_\Omega(z) = \lim_{\varepsilon\to 0} \frac{-1}{\pi} \int_{w\in\Omega,\ |z-w|>\varepsilon} \frac{1}{(z-w)^2}\,dm(w).$$

En utilisant des estimations a priori, dans cet article on résout le problème suivant de frontière libre : si $B\chi_\Omega$ appartient a l'espace de Sobolev $W^{\alpha,p}(\Omega)$ pour $0<\alpha\leqslant 1$, $1<p<\infty$ tels que $\alpha p>1$, alors le vecteur unitaire normal extérieur N sur $\partial\Omega$ est dans l'espace de Besov $B^{\alpha-1/p}_{p,p}(\partial\Omega)$.

Le résultat réciproque a été démontré récemment par Cruz et Tolsa. Nous avons donc que

$$B(\chi_\Omega) \in W^{\alpha,p}(\Omega) \quad\Longleftrightarrow\quad N \in B^{\alpha-1/p}_{p,p}(\partial\Omega).$$

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De cette équivalence et de quelques résultats récents de Cruz, Mateu et Orobitg, il résulte que la transformée de Beurling est bornée dans $W^{\alpha,p}(\Omega)$ si et seulement si le vecteur normal N appartient à $B^{\alpha-1/p}_{p,p}(\partial\Omega)$, en supposant que $\alpha p > 2$.

Keywords: Beurling transform; Lipschitz domain; Sobolev space; Besov space

1. Introduction

In this paper we show that the boundedness of the Beurling transform in the Sobolev spaces $W^{\alpha,p}(\Omega)$, with $0 < \alpha \leqslant 1$ and $1 < p < \infty$ such that $\alpha p > 1$, characterizes the Besov smoothness of the boundary $\partial\Omega$, whenever Ω is a $\mathcal{C}^1$ domain, or a Lipschitz domain "flat enough". This can be considered as a free boundary problem.

The Beurling transform of a function $f : \mathbb{C} \to \mathbb{C}$, with $f \in L^p$ for some $1 \leqslant p < \infty$, is defined by

$$Bf(z) = \lim_{\varepsilon\to 0} \frac{-1}{\pi} \int\limits_{|z-w|>\varepsilon} \frac{f(w)}{(z-w)^2}\,dm(w).$$

It is known that this limit exists a.e. The Beurling transform plays an essential role in the theory of quasiconformal mappings in the plane, because it intertwines the ∂ and $\bar\partial$ derivatives. Indeed, in the sense of distributions, one has

$$B(\bar\partial f) = \partial f.$$

Let $\Omega \subset \mathbb{C}$ be a bounded domain (open and connected). We say that $\Omega \subset \mathbb{C}$ is a (δ, R)-Lipschitz domain if for each $z \in \partial\Omega$ there exists a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$ with slope $\|A'\|_\infty \leqslant \delta$ such that, after a suitable rotation,

$$\Omega \cap B(z,R) = \bigl\{(x,y) \in B(z,R):\ y > A(x)\bigr\}.$$

If we do not care about the constants δ and R, then we just say that Ω is a Lipschitz domain. If in this definition we assume the function A to be of class $\mathcal{C}^1$, then we say that Ω is a $\mathcal{C}^1$ domain.

In [2] it has been shown that for any Lipschitz domain Ω and $0 < \alpha \leqslant 1$ and $1 < p < \infty$ such that $\alpha p > 1$, if the outward unit normal is in the Besov space $B^{\alpha-1/p}_{p,p}(\partial\Omega)$, then $B(\chi_\Omega)$ belongs to the Sobolev space $W^{\alpha,p}(\Omega)$. More precisely, the following estimate has been proved:

$$\bigl\|B(\chi_\Omega)\bigr\|_{\dot W^{\alpha,p}(\Omega)} \leqslant c\|N\|_{\dot B^{\alpha-1/p}_{p,p}(\partial\Omega)}, \tag{1.1}$$

where N stands for the outward normal unitary vector on $\partial\Omega$, $\dot W^{\alpha,p}(\Omega)$ is a homogeneous Sobolev space on Ω, and $\dot B^{\alpha-1/p}_{p,p}(\partial\Omega)$ is a homogeneous Besov space on $\partial\Omega$. See the next section for the precise definition of Sobolev and Besov spaces, as well as their homogeneous versions. The constant c in (1.1) may depend on p and on the Lipschitz character of Ω, i.e. on δ and on $\mathcal{H}^1(\partial\Omega)/R$ (here $\mathcal{H}^1$ stands for the length or 1-dimensional Hausdorff measure). Observe that, by the L^p boundedness of the Beurling transform,

$$\bigl\|B(\chi_\Omega)\bigr\|_{W^{\alpha,p}(\Omega)} \leqslant c\bigl(m(\Omega)^{1/p} + \bigl\|B(\chi_\Omega)\bigr\|_{\dot W^{\alpha,p}(\Omega)}\bigr).$$

Thus (1.1) guarantees that $B(\chi_\Omega) \in W^{\alpha,p}(\Omega)$ whenever $N \in \dot B^{\alpha-1/p}_{p,p}(\partial\Omega)$.

Our main result is a (partial) converse to (1.1):

Theorem 1.1. *Let $\Omega \subset \mathbb{C}$ be a (δ, R)-Lipschitz domain and $0 < \alpha \leqslant 1$ and $1 < p < \infty$ such that $\alpha p > 1$. If $\delta = \delta(p) > 0$ is small enough, then*

$$\|N\|_{\dot B^{\alpha-1/p}_{p,p}(\partial\Omega)} \leqslant c\bigl\|B(\chi_\Omega)\bigr\|_{\dot W^{\alpha,p}(\Omega)} + c\mathcal{H}^1(\partial\Omega)^{-\alpha+2/p}, \tag{1.2}$$

where c depends on δ and p.

Some remarks are in order. Notice first that $\mathcal{C}^1$ domains are (δ, R)-Lipschitz domains for every $\delta > 0$ and an appropriate $R = R(\delta)$. So the theorem applies to all $\mathcal{C}^1$ domains. Then, by combining the results from [2] with Theorem 1.1, one infers that, for a $\mathcal{C}^1$ domain Ω and α, p as above,

$$B(\chi_\Omega) \in W^{\alpha,p}(\Omega) \quad\Longleftrightarrow\quad N \in B^{\alpha-1/p}_{p,p}(\partial\Omega). \tag{1.3}$$