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Multi-valued, singular stochastic evolution inclusions

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Abstract

We provide an abstract variational existence and uniqueness result for multi-valued, monotone, non-coercive stochastic evolution inclusions in Hilbert spaces with general additive and Wiener multiplicative noise. As examples we discuss certain singular diffusion equations such as the stochastic 1-Laplacian evolution (total variation flow) in all space dimensions and the stochastic singular fast-diffusion equation. In case of additive Wiener noise we prove the existence of a unique weak-$*$ mean ergodic invariant measure.

Résumé

On propose un résultat abstrait variationnel d'existence et d'unicité pour les inclusions d'évolution stochastiques multivaluées, monotones et non coercitives dans les espaces de Hilbert avec un bruit général additif et un bruit de Wiener multiplicatif. À titre d'exemple, on discute les équations de la diffusion singulière particulières telle que l'évolution du 1-laplacien stochastique (le flux de la variation totale) dans toutes les dimensions d'espace et l'évolution de la diffusion rapide singulière stochastique. Dans le cas du bruit de Wiener additif, on démontre l'existence d'une mesure invariante faible-$*$ en moyenne ergodique.

MSC: 47D07; 60H15; 35R60

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1. Introduction

We consider the following evolution inclusion in a separable Hilbert space H

$$\begin{aligned} dX_t + A(t, X_t)\,dt &\ni dg_t, \quad t > 0, \\ X_0 &= x. \end{aligned} \tag{1.1}$$

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Here A is (among other assumptions) required to be a possibly multi-valued, singular, maximal monotone operator and g is a càdlàg path in H. The meaning of the expression dg_t will be specified below.

In particular, we are interested in stochastic evolution inclusions of the type

$$\begin{aligned} dX_t + A(t, X_t)\,dt &\ni dN_t, \quad t > 0, \\ X_0 &= x, \end{aligned} \tag{1.2}$$

where $\{N_t\}_{t\geqslant 0}$ is a càdlàg, adapted H-valued stochastic process on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ and in inclusions of the form

$$\begin{aligned} dX_t + A(t, X_t)\,dt &\ni B_t(X_t)\,dW_t, \quad t > 0, \\ X_0 &= x, \end{aligned} \tag{1.3}$$

where for some separable Hilbert space U, $B : [0,T] \times \Omega \times S \to L_2(U;H)$ takes values in the space of Hilbert–Schmidt operators from U to H and $\{W_t\}$ is a cylindrical Wiener process.

We prove the unique existence of solutions to (1.1)–(1.3) as well as the unique existence of a weak-$*$ mean ergodic invariant measure for (1.3) with additive noise, i.e. with $B_t(x) \equiv B_t$ independent of x.

We now comment on the main mathematical difficulties arising for singular evolution inclusions of type (1.1)–(1.3). The standard variational approach to (S)PDE of type (1.3) requires the drift operator A to be single-valued and to extend to a hemicontinuous, coercive operator $A : V \to V^*$ for some Gelfand triple $V \subseteq H \subseteq V^*$ (cf. [41,52,57]). The reflexivity of V and V^* is crucial for the construction of solutions. Therefore, the standard approach cannot be applied to highly singular (S)PDE such as the total variation flow, the two phase Stefan problem, plasma diffusion and the curve shortening flow. In all of these examples the space V degenerates in the sense that V or V^* fail to be reflexive. While recently increasing interest has been paid to this kind of singular, possibly multi-valued SPDE (cf. e.g. [6,12,16,24,59,66]), the unique existence of solutions could only be shown for additive noise and under strong dimensional restrictions. The principal idea of most of these works is the concept of (stochastic) evolution variational inequalities (EVI),[3] thus weakening the notion of solutions to (1.3). However, the approach via EVI has multiple drawbacks. First, it relies on the transformation of (1.3) into a random PDE and hence is restricted to simple structures of noise, such as additive or linear multiplicative noise. Second, due to the weaker notion of solutions it is hard to prove uniqueness. In fact, so far uniqueness of EVI could only be proven in case of sufficiently regular additive noise. Third, the construction of solutions to EVI still requires a coercivity condition of the type

$${}_{V^*}\langle A(u), u\rangle_V \geqslant c\|u\|_V^\alpha, \tag{1.4}$$

for some $\alpha \geqslant 1$, $c > 0$, which leads to restrictions on the dimension or the coercivity exponent α.

In order to remedy these obstacles for the class of equations considered in this paper, we introduce another Hilbert space S embedded compactly and densely into H such that

$$S \subseteq V \subseteq H \equiv H^* \subseteq V^* \subseteq S^*.$$

Subsequently, we will drop the intermediate space V and formulate the conditions of our hypotheses solely with respect to S. We assume that the drift A is maximal monotone and of at most linear growth in S^*. We are able to replace the strong coercivity assumption (1.4) by weak dissipativity in S formulated in an approximative way (cf. (A4) below).

Once existence and uniqueness of solutions for additive noise have been shown, limiting solutions for multiplicative noise may be constructed by "freezing the diffusion coefficients" and iteration methods. This is a well-known technique relying on the monotonicity of the drift (cf. e.g. [9,10]). However, the solutions constructed this way are in general not known to satisfy the actual SPDE in any way. By proving regularity properties (i.e. X_t taking values in S a.s.) we are able to identify the limiting solutions as variational solutions to (1.3) (cf. Definition 4.3 below).

Both existence and uniqueness of invariant measures for singular SPDE of the form (1.3) with additive noise are difficult problems in general. Since only in dimension $d = 1$ we expect a compact embedding of the energy space

[3] EVIs are also referred to as SVIs, that is, stochastic variational inequalities. We are following this naming convention in Appendix C.