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Characterization of ellipsoids through an overdetermined boundary value problem of Monge–Ampère type

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Abstract

The study of the optimal constant in an Hessian-type Sobolev inequality leads to a fully nonlinear boundary value problem, overdetermined with non-standard boundary conditions. We show that all the solutions have ellipsoidal symmetry. In the proof we use the maximum principle applied to a suitable auxiliary function in conjunction with an entropy estimate from affine curvature flow.

Résumé

L'étude de la meilleure constante dans une inégalité de Sobolev « de type hessien » conduit à un problème aux limites complètement non linéaire surdéterminé avec des conditions aux limites non standard. On démontre que les lignes de niveau de toutes les solutions de ce problème sont des ellipsoïdes. La démonstration utilise le principe du maximum pour une fonction auxiliaire, ainsi qu'une inégalité d'entropie pour le mouvement par courbure affine.

MSC: 35N25; 35B50; 35J60

Keywords: Overdetermined problems; Fully nonlinear equations; Curvature flow; Maximum principles

1. Introduction

In this paper we study the following fully nonlinear overdetermined boundary value problem

$$\begin{cases} \det D^2 u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ H_{n-1}|Du|^{n+1} = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^n$ is a smooth, bounded open set whose boundary has positive Gaussian curvature H_{n-1}, and c is a given positive constant. If we denote by ω_n the volume of the unit ball in $\mathbb{R}^n$ and Ω is any ellipsoid of measure $\omega_n c^{n/2}$, then

$$u(x) = \frac{|A(x-x_0)|^2 - c}{2} \tag{1.2}$$

is the solution to (1.1), for some $x_0 \in \mathbb{R}^n$ and some $n \times n$ matrix A with $\det A = 1$. Obviously $\Omega = \{x \in \mathbb{R}^n\colon |A(x-x_0)|^2 \leqslant c\}$. Our main result reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex, open set with C^2 boundary; a convex function $u \in C^2(\bar{\Omega})$ is a solution to problem (1.1) if and only if Ω is an ellipsoid of measure $\omega_n c^{n/2}$ and u has the specific form given in (1.2).*

In 1971 in a celebrated paper [38] Serrin proved that a smooth domain Ω is necessarily a ball if, for some constant $\gamma > 0$, there exists a solution $u \in C^2(\bar{\Omega})$ to the following problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = \gamma & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where ν is the unit outer normal to $\partial\Omega$. The main ingredients employed in the proof were a revisited Alexandrov moving plane method and a refinement of the maximum principle and Hopf's boundary point lemma. All such techniques soon became primary tools in the study of symmetries in PDE's (see for instance [21] and the references therein) when, in the wake of this pioneering paper, the study of overdetermined boundary value problems burst out.

Right after Serrin's paper the very same result was also obtained by Weinberger [45] with a very short proof. To better understand the key steps of our proof in the following sections, it is worth to briefly remind here the basic ideas behind Weinberger's one. First of all he showed that the auxiliary function $|Du|^2 - \frac{2}{n}u$ (being subharmonic in Ω) achieves its maximum γ^2 on the boundary of Ω. Then he observed that, in view of the Pohožaev identity, one has

$$\int_\Omega |Du|^2\,dx - \frac{2}{n}\int_\Omega u\,dx = \gamma^2|\Omega|$$

($|\Omega|$ denoting the measure of Ω), and he deduced that $|Du|^2 - \frac{2}{n}u$ is constant in Ω. This fact immediately carries the radial symmetry of the solution to (1.3).

Since these fundamental contributions, several alternative proofs and generalizations to linear and nonlinear operators followed (see for instance [45,29,16,26,10,23,5,6,20,7,11,19,22]). Maximum principle is always hidden somewhere in the proof, however some of the developed techniques do not require its explicit usage (we refer the interested reader to [5–7]).

Compared to most of the problems that can be found in literature, (1.1) has some unusual peculiarities. Firstly the differential operator is fully nonlinear, with strongly coupled second order derivatives. Secondly the problem admits both radially and non-radially-symmetric solutions. Such two features can be found in literature for instance in [25,31,33,26,34,3,13,2,17,4,18], where they rarely occur simultaneously and, to our knowledge, not for all dimensions.

The structure of our paper is the following. In Section 2 we introduce basic notation and preliminary results. Section 3 is the core of the paper and for the reader's convenience we split the proof in four claims. In the wake of Weinberger's paper we introduce an auxiliary function $\varphi(u, Du, D^2u)$ for which a maximum principle holds (see Claim 1 and Claim 2 below). In view of a Pohožaev type identity for Monge–Ampère equations we show that φ is constant in Ω (see Claim 3 below). Surprisingly, this provides informations on the evolution of $\partial\Omega$ by affine mean curvature flow. In particular, an equality sign is achieved in a fundamental entropy inequality (involving the affine surface area of Ω) which have been proved in [1] and as a consequence Ω turns out to be an ellipsoid (see Claim 4 below).

The use of the affine mean curvature machinery is somehow the most original idea in our proof. We observe that, at least in the planar case, such an idea is not needed (see [2]), and for completeness we sketch a different proof in Remark 3.1.

Now, before entering in the details of the proof of Theorem 1.1, we want to discuss the reasons which led us to consider the overdetermination in (1.1). They have to be found in connection with the study of Hessian Sobolev