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Full range of blow up exponents for the quintic wave equation in three dimensions [☆]

Joachim Krieger [a], Wilhelm Schlag [b]

[a] Bâtiment des Mathématiques, EPFL, Station 8, CH-1015 Lausanne, Switzerland

[b] Department of Mathematics, The University of Chicago, 5734 South University Avenue, Chicago, IL 60615, USA

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Abstract

For the critical focusing wave equation $\Box u = u^5$ on $\mathbb{R}^{3+1}$ in the radial case, we prove the existence of type II blow up solutions with scaling parameter $\lambda(t) = t^{-1-\nu}$ for all $\nu > 0$. This extends the previous work by the authors and Tataru where the condition $\nu > \frac{1}{2}$ had been imposed, and gives the optimal range of polynomial blow up rates in light of recent work by Duyckaerts, Kenig and Merle.

Résumé

On considère l'équation des ondes critique et focalisante $\Box u = u^5$ sur $\mathbb{R}^{3+1}$, dans le cas de solutions radiales. On montre l'existence de solutions explosives de type II avec $\lambda(t) = t^{-1-\nu}$, pour tout $\nu > 0$. Ce résultat est donc une généralisation des travaux des auteurs et de Tataru où on imposait une condition $\nu > \frac{1}{2}$, et donne l'intervalle optimale pour des solutions explosives de type II et dont la vitesse d'explosion est poiynomiale, à la lumière de résultats récents de Duyckaerts, Kenig et Merle.

Keywords: Energy critical wave equation; Finite time blow up; Dynamical rescaling

1. Introduction

We consider the energy critical focusing wave equation

$$\Box u = u^5, \qquad \Box = \partial_t^2 - \Delta \tag{1.1}$$

on $\mathbb{R}^{3+1}$, in the radial case. This equation has been intensely studied in a number of recent works: the remarkable series of papers [2–5] established a complete classification of all *possible* type II blow up dynamics, without proving their existence. In the works [7,1] a constructive approach to actually exhibit and thereby prove the existence of such type II dynamics was undertaken. Recall that a type II blow up solution $u(t,x)$ with blow up time T_* is one for which

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E-mail addresses: joachim.krieger@epfl.ch (J. Krieger), schlag@math.uchicago.edu (W. Schlag).

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$$\limsup_{t\to T_*}\left(\left\|u(t,\cdot)\right\|_{\dot H^1}+\left\|u_t(t,\cdot)\right\|_{L^2_x}\right)<\infty.$$

In [5], it is demonstrated that such solutions can be described as a sum of dynamically re-scaled ground states

$$\pm W(x)=\pm\left(1+\frac{|x|^2}{3}\right)^{-\frac{1}{2}}$$

plus a radiation term. In particular, for solutions where only one such bulk term is present, one can write the solution as

$$u(t,x)=W_{\lambda(t)}(x)+w(t,x)+o_{\dot H^1}(1),\qquad W_\lambda=\lambda^{\frac{1}{2}}W(\lambda x),\quad w(t,x)\in\dot H^1, \tag{1.2}$$

where the "error" $(w(t,\cdot),\partial_t w(t,\cdot))$ converges strongly in $\dot H^1\times L^2$ as $t\to T_*$, and we have the dynamic condition

$$\lim_{t\to T_*}(T_*-t)\lambda(t)=\infty. \tag{1.3}$$

In [7], it was shown that such solutions with $\lambda(t)=t^{-1-\nu}$ do exist, where $\nu>\frac{1}{2}$ is arbitrary. This left the question whether for *polynomial* rates the condition (1.3) is indeed optimal. Here we show that it is.

Theorem 1.1. *Let $\nu>0$ be given. Then there exists an energy class solution $u(t,x)$, which in fact has regularity $H^{1+\frac{\nu}{2}-}$, of the form (1.2), with*

$$\lambda(t)=t^{-1-\nu}.$$

Our method of proof is closely modeled on the construction from [7], of which we now recall the main steps:

(i) We write $u_0(t,r)=W_{\lambda(t)}(r)$ and iteratively modify u_0 in the form

$$u_{2k-1}=u_0+\sum_{j=1}^{2k-1}v_j$$

so that u_{2k-1} satisfies (1.1) up to an error of size t^N as $t\to 0+$; here N can be made as large as desired by taking k large, and the size is measured relative to the energy inside a light cone with tip at $r=0$, $t=0$.

(ii) We seek an exact solution via a perturbation: $u=u_{2k-1}+\varepsilon$. To solve for ε we switch to coordinates $R=\lambda(t)r$, $\tau=\int_t^\infty\lambda(s)\,ds=\frac{1}{\nu}t^{-\nu}$. The variable τ varies in the range $\tau_0\leqslant\tau<\infty$.

(iii) In the new coordinates, the driving linear operator is

$$\mathcal{L}=-\partial_{RR}-5W^4(R)\quad\text{on }(0,\infty).$$

We perform a spectral analysis of the operator, which exhibits a unique and simple negative eigenvalue, as well as continuum spectrum; in addition, there is a zero energy resonance. The latter renders the spectral measure singular at zero energy.

(iv) A contraction argument is set up for ε with a vanishing condition at $\tau=\infty$. For the contraction it is important that N in the first step is sufficiently large.

The first three steps in this paper are essentially the same as in [7]. It is in the final step that we improve on the procedure in [7]. In fact, in Proposition 2.8 of that paper the condition $\nu>\frac{1}{2}$ arises in order to make the embedding

$$\left(H^{1+2\alpha}\left(\mathbb{R}^3\right)\right)^5\subset H^{2\alpha}\left(\mathbb{R}^3\right)$$

for $\nu/2>2\alpha\geqslant\frac{1}{4}$. This was used to control the quintic terms in the construction of the *exact solution* via iteration and application of a suitable parametrix. In fact, the difference shall consist in a more detailed analysis of the *first iterate* for the exact solution, which we exhibit as a sum of two terms, one of which is smooth, the other of which satisfies a good L^∞-bound *near the origin*. This latter feature comes from the fact that the loss of smoothness of the approximate solution occurs precisely on the characteristic light cone, and thus one expects the exact solution to be smoother near the spatial origin.