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Regularity results for very degenerate elliptic equations

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Abstract

We consider a family of elliptic equations introduced in the context of traffic congestion. They have the form $\nabla \cdot (\nabla \mathcal{F}(\nabla u)) = f$, where $\mathcal{F}$ is a convex function which vanishes inside some convex set and is elliptic outside. Under some natural assumptions on $\mathcal{F}$ and f, we prove that the function $\nabla \mathcal{F}(\nabla u)$ is continuous in any dimension, extending a previous result valid only in dimension 2 (Santambrogio and Vespri, 2010 [14]).

Résumé

Dans cet article, on considère une famille d'équations elliptiques introduites dans le contexte d'un problème de transport congestionné. Ces équations sont de la forme $\nabla \cdot (\nabla \mathcal{F}(\nabla u)) = f$, où $\mathcal{F}$ est une fonction convexe qui vaut zéro sur un ensemble convexe et est uniformément elliptique au dehors de cet ensemble. Sous des conditions naturelles sur $\mathcal{F}$ et f, on démontre que la fonction $\nabla \mathcal{F}(\nabla u)$ est continue en toutes dimensions, ce qui étend un résultat précèdent en dimension 2 (Santambrogio et Vespri, 2010 [14]).

Keywords: Degenerate elliptic PDEs; Continuity of the gradient; Traffic congestion

1. Introduction

Given a bounded open subset Ω of $\mathbb{R}^n$, a convex function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$, and an integrable function $f : \Omega \to \mathbb{R}$, we consider a function $u : \Omega \to \mathbb{R}$ which locally minimizes the functional

$$\int_\Omega \mathcal{F}(\nabla u) + fu. \tag{1}$$

When $\nabla^2 \mathcal{F}$ is uniformly elliptic, namely there exist $\lambda, \Lambda > 0$ such that

$$\lambda \operatorname{Id} \leqslant \nabla^2 \mathcal{F} \leqslant \Lambda \operatorname{Id},$$

the regularity results of u in terms of $\mathcal{F}$ and f are well known.

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If $\mathcal{F}$ degenerates at only one point, then several results are still available. For instance, in the case of the p-Laplace equation with zero right-hand side, that is when $\mathcal{F}(v) = |v|^p$ and $f = 0$, the $C^{1,\alpha}$ regularity of u has been proved by Uraltseva [19], Uhlenbeck [18], and Evans [10] for $p \geqslant 2$, and by Lewis [13] and Tolksdorff [17] for $p > 1$ (see also [7,20]). Notice that in this case the equation is uniformly elliptic outside the origin.

More in general, one can consider functions whose degeneracy set is a convex set: for example, for $p > 1$ one may consider

$$\mathcal{F}(v) = \frac{1}{p}\big(|v| - 1\big)_+^p \quad \forall v \in \mathbb{R}^n, \tag{2}$$

so that the degeneracy set is the entire unit ball. There are many Lipschitz results on u in this context [11,9,2], and in general no more regularity than L^∞ can be expected on ∇u. Indeed, when $\mathcal{F}$ is given by (2) and f is identically 0, every 1-Lipschitz function solves the equation. However, as proved in [14] in dimension 2, something more can be said about the regularity of $\nabla\mathcal{F}(\nabla u)$, since either it vanishes or we are in the region where the equation is more elliptic.

The problem of minimizing the energy (1) with the particular choice of $\mathcal{F}$ given in (2) arises in the context of traffic congestion. Indeed, it is equivalent to the problem

$$\min\left\{\int_\Omega |\sigma| + \frac{1}{p'}|\sigma|^{p'}:\ \sigma \in L^{p'}(\Omega), \nabla\cdot\sigma = f, \sigma\cdot\nu_{\partial\Omega} = 0\right\}, \tag{3}$$

where p' satisfies $1/p + 1/p' = 1$, and σ represents the traffic flow. The particular choice of $\mathcal{F}$, or equivalently of its convex conjugate $\mathcal{F}^*$ which appears in (3) as an integrand, satisfies two demands: $\mathcal{F}^*$ has more than linear growth at infinity (so to avoid "congestion") and satisfies $\liminf_{w\to 0}|\nabla\mathcal{F}^*(w)| > 0$ (which means that moving in an empty street has a nonzero cost). As shown in [3], the unique optimal minimizer $\bar\sigma$ turns out to be exactly $\nabla\mathcal{F}(\nabla u)$, where $\mathcal{F}$ is defined by (2).

In this paper we prove that, if $\mathcal{F}$ vanishes on some convex set E and is elliptic outside such a set, then $\mathcal{H}(\nabla u)$ is continuous for any continuous function $\mathcal{H}: \mathbb{R}^n \to \mathbb{R}$ which vanishes on E. In particular, by applying this result with $\mathcal{H} = \partial_i\mathcal{F}$ $(i = 1, \dots, n)$ where $\mathcal{F}$ is as in (2), our continuity result implies that $\bar\sigma = \nabla\mathcal{F}(\nabla u)$ (the minimizer of (3)) is continuous in the interior of Ω. This result is important for the following reason: as shown in [5] (see also [3]), one can build a measure on the space of possible paths starting from $\bar\sigma$, and this optimal traffic distribution satisfies a Wardrop equilibrium principle: no traveler wants to change his path, provided all the others keep the same strategy. In other words, every path which is followed by somebody is a geodesics with respect to the metric $g(\bar\sigma(x))\,\mathrm{Id}$ (where $g(t) = 1 + t^{p-1}$ is the so-called "congestion function"), which is defined in terms of the traffic distribution itself. Hence, our continuity result shows that the metric is continuous (so, in particular, well defined at every point), which allows to set and study the geodesic problem in the usual sense.

Since we want to allow any bounded convex set as degeneracy set for $\mathcal{F}$, before stating the result we introduce the notion of norm associated to a convex set, which is used throughout the paper to identify the nondegenerate region. Given a bounded closed convex set $E \subseteq \mathbb{R}^n$ such that 0 belongs to $\mathrm{Int}(E)$ (the interior of E), and denoting by tE the dilation of E by a factor t with respect to the origin, we define $|\cdot|_E$ as

$$|e|_E := \inf\{t > 0:\ e \in tE\}. \tag{4}$$

Notice that $|\cdot|_E$ is a convex positively 1-homogeneous function. However $|\cdot|_E$ is not symmetric unless E is symmetric with respect to the origin.

The main result of the paper proves that, in the context introduced before, $\nabla\mathcal{F}(\nabla u)$ is continuous.

Theorem 1.1. *Let n be a positive integer, Ω a bounded open subset of $\mathbb{R}^n$, $f \in L^q(\Omega)$ for some $q > n$. Let E be a bounded, convex set with $0 \in \mathrm{Int}(E)$. Let $\mathcal{F}: \mathbb{R}^n \to \mathbb{R}$ be a convex nonnegative function such that $\mathcal{F} \in C^2(\mathbb{R}^n \setminus \bar E)$. Let us assume that for every $\delta > 0$ there exist $\lambda_\delta, \Lambda_\delta > 0$ such that*

$$\lambda_\delta I \leqslant \nabla^2\mathcal{F}(x) \leqslant \Lambda_\delta I \quad \text{for a.e. } x \text{ such that } 1 + \delta \leqslant |x|_E \leqslant 1/\delta. \tag{5}$$

Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional