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Stochastic homogenization and random lattices

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Abstract

We present some variants of stochastic homogenization theory for scalar elliptic equations of the form $-\operatorname{div}[A(\frac{x}{\varepsilon},\omega)\nabla u(x,\omega)] = f$. These variants basically consist in defining stochastic coefficients $A(\frac{x}{\varepsilon},\omega)$ from stochastic deformations (using random diffeomorphisms) of the periodic setting, as announced in [X. Blanc, C. Le Bris, P.-L. Lions, Une variante de la théorie de l'homogénéisation stochastique des opérateurs elliptiques (A variant of stochastic homogenization theory for elliptic operators), C. R. Acad. Sci. Sér. I 343 (2006) 717–727]. The settings we define are not covered by the existing theories. We also clarify the relation between this type of questions and our construction, performed in [X. Blanc, C. Le Bris, P.-L. Lions, A definition of the ground state energy for systems composed of infinitely many particles, Commun. Partial Differential Equations 28 (1–2) (2003) 439–475; X. Blanc, C. Le Bris, P.-L. Lions, The energy of some microscopic stochastic lattices, Arch. Rat. Mech. Anal. 184 (2) (2007) 303–339], of the energy of, both deterministic and stochastic, microscopic infinite sets of points in interaction.

Résumé

Nous présentons dans cet article quelques variantes de la théorie de l'homogénéisation stochastique pour les équations elliptiques scalaires de la forme $-\operatorname{div}[A(\frac{x}{\varepsilon},\omega)\nabla u(x,\omega)] = f$. Ces variantes consistent essentiellement à définir les coefficients $A(\frac{x}{\varepsilon},\omega)$ comme déformations stochastiques (par des difféomorphismes aléatoires) de coefficients périodiques. Ce travail a été annoncé dans [X. Blanc, C. Le Bris, P.-L. Lions, Une variante de la théorie de l'homogénéisation stochastique des opérateurs elliptiques (A variant of stochastic homogenization theory for elliptic operators), C. R. Acad. Sci. Sér. I 343 (2006) 717–727]. Les cas que nous définissons ainsi ne sont pas inclus dans les théories existantes de l'homogénéisation stochastique. Nous établissons également un lien entre ce type de problème et celui d'une définition de l'énergie moyenne d'un système infini de particules, que nous avons traité dans [X. Blanc, C. Le Bris, P.-L. Lions, A definition of the ground state energy for systems composed of infinitely many particles, Commun. Partial Differential Equations 28 (1–2) (2003) 439–475] pour le cas déterministe, et dans [X. Blanc, C. Le Bris, P.-L. Lions, The energy of some microscopic stochastic lattices, Arch. Rat. Mech. Anal. 184 (2) (2007) 303–339] pour le cas stochastique.

Keywords: Stochastic homogenization; Thermodynamic limit; Random lattices

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1. Introduction

We study homogenization for scalar elliptic equations in divergence form with random coefficients:

$$-\operatorname{div}\left[A_\varepsilon(x,\omega)\nabla u(x,\omega)\right] = f. \tag{1.1}$$

In this context, the purpose of this article is two-fold.

First, we aim at slightly extending the usual ergodic stationary setting (see for instance [2,7]) by considering specific cases of random coefficients $A_\varepsilon(x,\omega)$, mainly of the form:

$$A_\varepsilon(x,\omega) = A\left(\frac{x}{\varepsilon},\omega\right), \tag{1.2}$$

not covered by the existing theories. These coefficients are typically obtained using random deformations of periodic coefficients. A prototypical case of such coefficients reads:

$$A\left(\frac{x}{\varepsilon},\omega\right) = A_{\mathrm{per}}\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right)\right), \tag{1.3}$$

where A_{per} is $\mathbb{Z}^d$-periodic, and Φ is almost surely a diffeomorphism. Its gradient $\nabla\Phi$ is assumed stationary in the sense,

$$\forall k \in \mathbb{Z}^d, \quad \nabla\Phi(x+k,\omega) = \nabla\Phi(x,\tau_k\omega) \text{ almost everywhere in } x, \text{ almost surely,}$$

for a certain ergodic group action τ. Note that, although this sounds as a special case of existing theories, it is not. The above setting has been introduced in our previous work [4], and is recalled in details in Sections 1.2 and 1.3. Several variants along this general line are examined here, in Sections 2 and 5. We show that all these variants allow for *explicit* homogenization results. That is, we are able to prove that homogenization holds and identify the homogenized limit, using corrector problems, which are shown to be well-posed. A specific case (developed in Section 3) is that of a diffeomorphism Φ in (1.3) that is a "small" perturbation of Identity. Then, using a Taylor expansion with respect to a small parameter measuring the perturbation, we are able to show that this specific stochastic homogenization setting reduces to some particular, new, situation of *periodic* homogenization.

Our second purpose is to clarify the relation between the above questions of homogenization theory and our long term endeavour to define the energy of an infinite set of point particles in interaction, as exposed in [3,5]. The reader is likely to be less familiar with that latter problem than with the classical homogenization problem. So, let us recall it briefly. More will be said in Section 4. If we are given an infinite set of points x_i, say interacting with the two-body potential $W(x_i - x_j)$, it is an easy exercise to define the notion of *energy per particle* of this assembly of particles when the x_i are periodically arranged. Some slight extensions of periodicity, such as quasi-periodicity, may also be treated. The construction also applies to energy models more sophisticated than the two-body interaction chosen here for simplicity of exposition. We will not enter the details of such questions, which have been the subject of many publications of ours (and others) in the past years. On the other hand, when the positions of the particles are more general, defining the energy per particle is a challenging question. In [3,5], we addressed that latter question, respectively for some "general" deterministic sets of points, and for sets of random points. We will return to this in Section 4. The point was to determine the appropriate geometric properties that allow for defining the energy. It turns out that the properties we exhibited for that purpose have their counterpart in homogenization theory. This is what we are going to show in Sections 4 and 5 of the present work. In the language of homogenization, the positions x_i of the point particles may intuitively be thought of as the *obstacles*, or, equivalently, the vertices of the unit cells. More mathematically, the positions x_i may be used to define, using a construction introduced in [3], an appropriate algebra of functions, namely the smallest algebra, closed for some uniform norm on $\mathbb{R}^d$ (say L^∞), containing functions of the form,

$$a(x) = \sum_{i\in\mathbb{N}} \varphi(x - x_i),$$

with, say, $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Taking the entries A_{ij} of the matrix A in (1.2) in this algebra, one may then ask the question of the homogenization of (1.1) within this algebra. Using this construction, we establish a correspondence between the homogenization problem and the, apparently distant, problem of definition of energies for sets of point particles.