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Derivatives of (modified) Fredholm determinants and stability of standing and traveling waves ☆

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Abstract

Continuing a line of investigation initiated in [F. Gesztesy, Y. Latushkin, K.A. Makarov, Evans functions, Jost functions, and Fredholm determinants, Arch. Rat. Mech. Anal. 186 (2007) 361–421] exploring the connections between Jost and Evans functions and (modified) Fredholm determinants of Birman–Schwinger type integral operators, we here examine the stability index, or sign of the first nonvanishing derivative at frequency zero of the characteristic determinant, an object that has found considerable use in the study by Evans function techniques of stability of standing and traveling wave solutions of partial differential equations (PDE) in one dimension. This leads us to the derivation of general perturbation expansions for analytically-varying modified Fredholm determinants of abstract operators. Our main conclusion, similarly in the analysis of the determinant itself, is that the derivative of the characteristic Fredholm determinant may be efficiently computed from first principles for integral operators with semi-separable integral kernels, which include in particular the general one-dimensional case, and for sums thereof, which appears to offer applications in the multi-dimensional case.

A second main result is to show that the multi-dimensional characteristic Fredholm determinant is the renormalized limit of a sequence of Evans functions defined in [G.J. Lord, D. Peterhof, B. Sandstede, A. Scheel, Numerical computation of solitary waves in infinite cylindrical domains, SIAM J. Numer. Anal. 37 (2000) 1420–1454] on successive Galerkin subspaces, giving a natural extension of the one-dimensional results of [F. Gesztesy, Y. Latushkin, K.A. Makarov, Evans functions, Jost functions, and Fredholm determinants, Arch. Rat. Mech. Anal. 186 (2007) 361–421] and answering a question of [J. Niesen, Evans function calculations for a two-dimensional system, presented talk, SIAM Conference on Applications of Dynamical Systems, Snowbird, UT, USA, May 2007] whether this sequence might possibly converge (in general, no, but with renormalization, yes). Convergence is useful in practice for numerical error control and acceleration.

Résumé

Nous poursuivons l'étude, commencée dans [F. Gesztesy, Y. Latushkin, K.A. Makarov, Evans functions, Jost functions, and Fredholm determinants, Arch. Rat. Mech. Anal. 186 (2007) 361–421], des liens entre les fonctions de Jost et d'Evans et les déterminants (modifiés) de Fredholm d'opérateurs intégraux de type Birman–Schwinger. Nous examinons ici l'indice de stabilité,

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c'est-à-dire le signe de la première dérivée non nulle, à la fréquence zéro, du déterminant caractéristique. Cet indice a trouvé une application importante dans l'étude, par des techniques de fonction d'Evans, des la stabilité des solutions de type ondes progressives de systèmes d'équations aux dérivées partielles en une dimension d'espace. Cela nous amène à écrire des formules de développements généraux de type perturbatif pour les déterminants modifiés de Fredholm d'opérateurs analytiques abstraits. Notre conclusion principale est que la dérivée du déterminant caractéristique de Fredholm, comme le déterminant lui-même, peut être calculée efficacement pour des opérateurs intégraux dont les noyaux sont semiséparables et pour les sommes. La première classe d'opérateurs inclut en particulier le cas général en une dimension d'espace ; ce dernier laisse envisager des applications au cas multidimensionnel.

Le deuxième résultat principal est la démonstration que le déterminant caractéristique multidimensionnel de Fredholm est la limite renormalisée d'une suite de fonctions d'Evans, définie dans [G.J. Lord, D. Peterhof, B. Sandstede, A. Scheel, Numerical computation of solitary waves in infinite cylindrical domains, SIAM J. Numer. Anal. 37 (2000) 1420–1454], sur des sous-espaces emboîtés de Galerkin ; ce résultat est une extension naturelle des résultats unidimensionnels de [F. Gesztesy, Y. Latushkin, K.A. Makarov, Evans functions, Jost functions, and Fredholm determinants, Arch. Rat. Mech. Anal. 186 (2007) 361–421], et répond à la question, posée dans [J. Niesen, Evans function calculations for a two-dimensional system, presented talk, SIAM Conference on Applications of Dynamical Systems, Snowbird, UT, USA, May 2007], de la convergence de cette suite (la réponse est qu'en général il n'y a pas convergence, mais qu'on peut obtenir la convergence après renormalisation). La convergence est utile dans la pratique pour le contrôle des erreurs et l'accélération des calculs numériques.

MSC: primary 35J10, 45P05; secondary 35P05, 47B10

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1. Introduction

A problem of general interest is to determine the spectrum of a general variable-coefficient linear differential operator $L = \sum_{|\alpha|=0}^{N} a_\alpha(x)\partial_x^\alpha$, $a_\alpha(x) \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^d$, with prescribed behavior of the coefficients as $|x| \to \infty$. This arises naturally, for example, in the study of traveling- or standing-wave solutions of nonlinear PDEs in a wide variety of applications, as described, for example, in the survey articles [35,47], and the references therein. In the one-dimensional case, $d = 1$, a very useful and general tool for this purpose is the *Evans function* [1,6–9,30], defined as a Wronskian $\mathcal{E}(z)$ of bases of the set of solutions $\Psi_\pm$ of the associated eigenvalue ODE $(L-\lambda)\Psi = 0$ decaying at $x = +\infty$ and $x = -\infty$, respectively, whose zeros correspond in location and multiplicity with the eigenvalues of L.

Among the many applications of the Evans function, perhaps the simplest and most general is the computation of the *stability index*,

$$\Gamma = \operatorname{sgn}\big(d_z^k \mathcal{E}(0)\big)\operatorname{sgn}\big(\mathcal{E}(+\infty)\big), \tag{1.1}$$

whose sign determines the parity of the number of unstable eigenvalues λ, or eigenvalues with positive real part $\operatorname{Re}(\lambda) > 0$, where $d_z^k \mathcal{E}(0)$ is the first nonvanishing derivative of $\mathcal{E}(z)$ at $z = 0$; see, for example, [9,30,47]. (A standard property of the Evans function is that it may be constructed so as to respect complex conjugation; in particular, it may be taken real-valued for $z \in \mathbb{R}$.) A problem that has received considerable recent interest[1] is to extend the Evans function, and in particular the stability index, to the more general setting of multi-dimensions in a way that is useful for practical computations. Here, we refer mainly to numerical computation, as presumably the only feasible way to treat large-scale problems associated with multi-dimensions.

Various different constructions have been suggested toward this end; see, for example, [3,4,23]. However, only one of these, the Galerkin approximation method of [23] (described in Section 4), seems in principle computable, and the computations involved appear quite numerically intensive. (So far, no such computations have satisfactorily been carried out, though, see the proposed methods discussed in [18,27].) It is therefore highly desirable to explore other directions that may be more computationally efficient.

Here, we follow a very natural direction first proposed in [11]. Specifically, it is shown in [11] for a quite general class of one-dimensional operators L that the Evans function, appropriately normalized, agrees with a (modified)

[1] For example, this was a focus topic of the workshop "Stability Criteria for Multi-Dimensional Waves and Patterns", at the American Institute of Mathematics (AIM) in Palo Alto (CA, USA), May 16–20, 2005.