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A nonlocal p-Laplacian evolution equation with Neumann boundary conditions

F. Andreu [a,*], J.M. Mazón [b], J.D. Rossi [c,d], J. Toledo [b]

[a] *Departament de Matemàtica Aplicada, Universitat de València, Valencia, Spain*
[b] *Departament d'Anàlisi Matemàtica, Universitat de València, Valencia, Spain*
[c] *IMDEA Matematicas, C-IX, Universitat Autonoma, Campus Cantoblanco, Madrid, Spain*
[d] *Departamento de Matemática, FCEyN UBA (1428), Buenos Aires, Argentina*

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Abstract

In this paper we study the nonlocal p-Laplacian type diffusion equation,

$$u_t(t,x) = \int_{\Omega} J(x-y)\big|u(t,y)-u(t,x)\big|^{p-2}\big(u(t,y)-u(t,x)\big)\,dy.$$

If $p>1$, this is the nonlocal analogous problem to the well-known local p-Laplacian evolution equation $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with homogeneous Neumann boundary conditions. We prove existence and uniqueness of a strong solution, and if the kernel J is rescaled in an appropriate way, we show that the solutions to the corresponding nonlocal problems converge strongly in $L^\infty(0,T;L^p(\Omega))$ to the solution of the p-Laplacian with homogeneous Neumann boundary conditions. The extreme case $p=1$, that is, the nonlocal analogous to the total variation flow, is also analyzed. Finally, we study the asymptotic behavior of the solutions as t goes to infinity, showing the convergence to the mean value of the initial condition.

Résumé

Dans cet article, on étudie l'équation de diffusion non locale de type p-laplacien

$$u_t(t,x) = \int_{\Omega} J(x-y)\big|u(t,y)-u(t,x)\big|^{p-2}\big(u(t,y)-u(t,x)\big)\,dy.$$

Si $p>1$, elle constitue le problème non local associé à l'équation d'évolution avec l'opérateur p-laplacien local $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ et avec des conditions aux limites de type Neumann homogène. On montre l'existence et l'unicité de la solution forte, et moyennant un changement d'échelle approprié sur le noyau J, on montre que la solution du problème non local converge fortement dans $L^\infty(0,T;L^p(\Omega))$ vers la solution du problème local avec des conditions aux limites de type Neumann homogène. On analyse aussi le cas limite $p=1$ qui correspond à l'équation non locale correspondant au problème de calcul de variation totale. Finalement, on étudie le comportement asymptotique de la solution lorsque $t\to\infty$, et on montre que la solution converge vers la moyenne de la donnée initiale.

* Corresponding author.
E-mail addresses: fuensanta.andreu@uv.es (F. Andreu), mazon@uv.es (J.M. Mazón), jrossi@dm.uba.ar (J.D. Rossi), toledojj@uv.es (J. Toledo).

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1. Introduction and presentation of results

Our main goal in this paper is to study the following nonlocal nonlinear diffusion problem, which we call the *nonlocal p-Laplacian problem* (with homogeneous Neumann boundary conditions),

$$P_p^J(u_0) \quad \begin{cases} u_t(t,x) = \displaystyle\int_\Omega J(x-y)\big|u(t,y)-u(t,x)\big|^{p-2}\big(u(t,y)-u(t,x)\big)\,dy, \\ u(x,0) = u_0(x). \end{cases}$$

Here $J:\mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)\,dx = 1$ (this last condition is not necessary to prove our results, it is imposed to simplify the exposition), $1 \leqslant p < +\infty$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain.

Nonlocal evolution equations of the form:

$$u_t(t,x) = J * u - u(t,x) = \int_{\mathbb{R}^N} J(x-y)\big(u(t,y)-u(t,x)\big)\,dy, \tag{1.1}$$

and variations of it, have been recently widely used to model diffusion processes, see [7–9,15–17,19,22,23,26,28] and [31]. Moreover, nonlocal problems of type $P_p^J(u_0)$ have been used recently in the study of deblurring and denoising of images (see [24]).

As stated in [22], if $u(t,x)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then the convolution $(J*u)(t,x) = \int_{\mathbb{R}^N} J(y-x)u(t,y)\,dy$ is the rate at which individuals are arriving to position x from all other places and $-u(t,x) = -\int_{\mathbb{R}^N} J(y-x)u(t,x)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies Eq. (1.1).

Eq. (1.1) is called a nonlocal diffusion equation since the diffusion of the density u at a point x and time t does not only depend on $u(t,x)$, but on all the values of u in a neighborhood of x through the convolution term $J*u$. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed [22]. However, there is no regularizing effect in general (see [16]).

When dealing with local evolution equations, two models of nonlinear diffusion has been extensively studied in the literature, the porous medium equation, $u_t = \Delta u^m$, and the p-Laplacian evolution, $u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. In the first case (for the porous medium equation) a nonlocal analogous equation was studied in [7] (see also [18]). Our main objective in this paper is to study the nonlocal equation P_p^J, that is, the nonlocal analogous to the p-Laplacian evolution.

Concerning boundary conditions for nonlocal problems, if, instead of (1.1), we look at

$$u_t(t,x) = \int_\Omega J(x-y)\big(u(t,y)-u(t,x)\big)\,dy,$$

the right-hand side takes into account the diffusion inside the domain Ω. In fact, as we have explained, the integral $\int J(x-y)(u(t,y)-u(t,x))\,dy$ takes into account the individuals arriving or leaving position x from or to other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. There is no flux of individuals across the boundary. This is the analogous of what is called homogeneous Neumann boundary conditions in the literature. In this sense, problem $P_p^J(u_0)$ has to be seen as a problem with homogeneous Neumann boundary condition. For $p = 2$, in [20] (see also [19]) it is proved that solutions to the linear problem $P_2^J(u_0)$ converge to the