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Integral inequalities for the Hilbert transform applied to a nonlocal transport equation

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Abstract

We prove several weighted inequalities involving the Hilbert transform of a function $f(x)$ and its derivative. One of those inequalities,

$$-\int \frac{f_x(x)[Hf(x)-Hf(0)]}{|x|^\alpha}\,\mathrm{d}x \geqslant C_\alpha \int \frac{(f(x)-f(0))^2}{|x|^{1+\alpha}}\,\mathrm{d}x,$$

is used to show finite time blow-up for a transport equation with nonlocal velocity.

Résumé

Dans cet article nous présentons la démonstration de plusieurs inegalités satisfaites par la transformation de Hilbert d'une fonction $f(x)$ et sa derivée $f'(x)$. Nous avons l'estimation suivante :

$$-\int \frac{f_x(x)[Hf(x)-Hf(0)]}{|x|^\alpha}\,\mathrm{d}x \geqslant C_\alpha \int \frac{(f(x)-f(0))^2}{|x|^{1+\alpha}}\,\mathrm{d}x,$$

où la constante C_α est strictement positive. Nous avons aussi utilisé cette inegalité pour démontrer l'explosion en un temps fini des solutions d'une équation de transport avec une vitesse nonlocale.

MSC: 35Q35; 76B03; 42A38; 35L67

Keywords: Integral inequalities; Hilbert transform; Transport equations

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1. Introduction

In this paper we show the existence of finite-time singularities for a Burgers type equation with nonlocal velocity in one space variable,

$$f_t - (Hf)f_x = 0, \tag{1}$$

where $H(\cdot)$ denotes the Hilbert transform (see (2) below), improving the results of our previous paper [4] in the sense that we can obtain here blow-up for a much wider class of initial data. One motivation for the study of that equation is its analogy with both, the 2D quasi-geostrophic equation and the 3D Euler in vorticity form (see [2,3,5,6,8–10] and [11] for a variety of 1D models involving nonlocal operators, the majority of them have their origin in the seminal paper of Constantin, Lax and Majda [3]). For instance the 2D quasi-geostrophic equation belongs to the class of transport equations with nonlocal velocities:

$$\theta_t - R_2(\theta)\frac{\partial\theta}{\partial x_1} + R_1(\theta)\frac{\partial\theta}{\partial x_2} = 0,$$

where the velocity is $u = (-R_2(\theta), R_1(\theta))$ and $R(\theta) = (R_1(\theta), R_2(\theta))$ are the Riesz transforms of θ. This is a well-known model for the dynamic of mixtures of cold and hot air.

Another motivation for (1) is its similarity in structure with the Birkhoff–Rott equation for the evolution of vortex sheets (see [1,4] and [7]).

In this paper we present examples showing the existence of such singularities, our proofs below will take advantage of certain weighted norm inequalities satisfied by the Hilbert transform. There is a vast literature about such topic, but, in our case, we need very precise estimates, separately for both even and odd functions, which do not follow directly from the general theorems. Since the weights involved are powers of the independent variable, the Mellin transform is a very adequate instrument to produce those sharp estimates.

In the following we shall consider the Hilbert transform defined by the formulas:

$$Hf(x) = \frac{1}{\pi}\,\mathrm{PV}\int \frac{f(y)}{x-y}\,\mathrm{d}y, \tag{2}$$

or $\widehat{Hf}(\xi) = -\mathrm{i}\,\mathrm{sign}(\xi)\hat{f}(\xi)$, where $\hat{f}(\xi) = \int \mathrm{e}^{-2\pi \mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x$ denotes the Fourier transform.

We shall make use also of the identity,

$$\int_0^\infty t^{-\beta}\frac{\mathrm{d}t}{t-1} = \pi\cot(\pi\beta), \tag{3}$$

valid for $0 < \mathrm{Re}(\beta) < 1$ and the explicit formula:

$$H\big(|x|^\alpha\big) = -\frac{1}{\pi}\tan\left(\frac{\alpha\pi}{2}\right)\mathrm{sign}(x)|x|^\alpha,$$

with $-3 < \alpha < 1$ and $\alpha \neq 0, -1, -2$. The distribution $1/|x|^\beta$ is defined by,

$$\frac{1}{|x|^\beta}(\varphi) = \mathrm{PV}\int \frac{\varphi(x) - \varphi(0)}{|x|^\beta}\,\mathrm{d}x \quad \text{if } 1 < \beta < 2,$$

and by,

$$\frac{1}{|x|^\beta}(\varphi) = \mathrm{PV}\int \frac{\varphi(x) - \varphi(0) - x\varphi_x(0)}{|x|^\beta}\,\mathrm{d}x \quad \text{if } 2 < \beta < 3.$$

To present our results we find it convenient to introduce the following functional spaces: For $0 < \alpha < 2$, $H^1_\alpha(\mathbb{R})$ is the closure of $\mathcal{C}^1_0(\mathbb{R})$ under the norm:

$$\|f\|^2 = \|f\|^2_{L^\infty} + \int \frac{(f(x) - f(0))^2}{|x|^{1+\alpha}}\,\mathrm{d}x + \int \frac{(f_x(x))^2}{|x|^{\alpha-1}}\,\mathrm{d}x,$$

and for $\alpha \leqslant 0$ we define $H^1_\alpha(\mathbb{R})$ as before taking out the correction term $f(0)$.

We will prove the following inequalities involving Hilbert transforms: