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The Hardy inequality and the heat equation in twisted tubes

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Abstract

We show that a twist of a three-dimensional tube of uniform cross-section yields an improved decay rate for the heat semigroup associated with the Dirichlet Laplacian in the tube. The proof employs Hardy inequalities for the Dirichlet Laplacian in twisted tubes and the method of self-similar variables and weighted Sobolev spaces for the heat equation.

Résumé

Nous montrons que la torsion d'un tube non-borné à section transversale constante dans l'espace euclidien tridimensionnel induit une amélioration du taux de décroissance pour le semi-groupe associé à l'equation de la chaleur avec des conditions aux limites de Dirichlet dans le tube. La démonstration utilise des inégalités de type Hardy pour le laplacien Dirichlet dans les tubes torsadés et la méthode de variables de similarité et les espaces de Sobolev à poids gaussiens pour l'équation de la chaleur.

Keywords: Twisted tubes; Hardy inequality; Dirichlet Laplacian; Heat equation; Large time behaviour of solutions

1. Introduction

It has been shown recently in [7] that a local twist of a straight three-dimensional tube $\Omega_0 := \mathbb{R} \times \omega$ of non-circular cross-section $\omega \subset \mathbb{R}^2$ leads to an effective repulsive interaction in the Schrödinger equation of a quantum particle constrained to the twisted tube Ω_θ. More precisely, there is a Hardy-type inequality for the particle Hamiltonian modelled by the Dirichlet Laplacian $-\Delta_D^{\Omega_\theta}$ at its threshold energy E_1 if, and only if, the tube is twisted (cf. Fig. 1). That is, the inequality,

$$-\Delta_D^{\Omega_\theta} - E_1 \geqslant \varrho, \tag{1.1}$$

holds true, in the sense of quadratic forms in $L^2(\Omega_\theta)$, with a positive function ϱ provided that the tube is twisted, while ϱ is necessarily zero for Ω_0. Here E_1 coincides with the first eigenvalue of the Dirichlet Laplacian $-\Delta_D^\omega$ in the cross-section ω.

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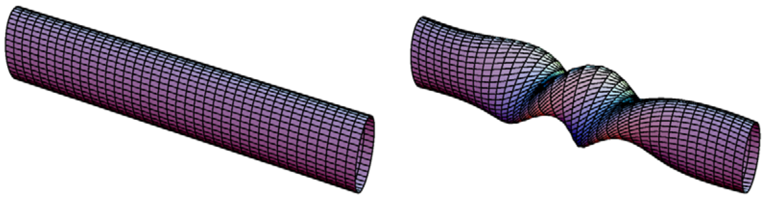

Fig. 1. Untwisted and twisted tubes of elliptical cross-section.

The inequality (1.1) has important consequences for conductance properties of quantum waveguides. It clearly implies the absence of bound states (i.e., stationary solutions to the Schrödinger equation) below the energy E_1 even if the particle is subjected to a small attractive interaction, which can be either of potential or geometric origin (cf. [7] for more details). At the same time, a repulsive effect of twisting on eigenvalues embedded in the essential spectrum has been demonstrated in [14]. Hence, roughly speaking, the twist prevents the particle to be trapped in the waveguide. Additional spectral properties of twisted tubes have been studied in [9,18,2].

It is natural to ask whether the repulsive effect of twisting demonstrated in [7] in the quantum context has its counterpart in other areas of physics, too. The present paper gives an affirmative answer to this question for systems modeled by the diffusion equation in the tube Ω_θ:

$$u_t - \Delta u = 0, \tag{1.2}$$

subject to Dirichlet boundary conditions on $\partial\Omega_\theta$. Indeed, we show that the twist is responsible for a faster convergence of the solutions of (1.2) to the (zero) stable equilibrium. The second objective of the paper is to give a new (simpler and more direct) proof of the Hardy inequality (1.1) under weaker conditions than those in [7].

1.1. The main result

Before stating the main result about the large time behavior of the solutions to (1.2), let us make some comments on the subtleties arising with the study of the heat equation in Ω_θ.

The specific deformation Ω_θ of Ω_0 via twisting we consider can be visualized as follows: instead of simply translating ω along $\mathbb{R}$ we also allow the (non-circular) cross-section ω to rotate with respect to a (non-constant) angle $x_1 \mapsto \theta(x_1)$. See Fig. 1 (the precise definition is postponed until Section 2, cf. Definition 2.1). We assume that the deformation is local, i.e.,

$$\dot\theta \text{ has compact support in } \mathbb{R}. \tag{1.3}$$

Then the straight and twisted tubes have the same spectrum (cf. [17, Sec. 4]):

$$\sigma\big(-\Delta_D^{\Omega_\theta}\big) = \sigma_{\mathrm{ess}}\big(-\Delta_D^{\Omega_\theta}\big) = [E_1, \infty). \tag{1.4}$$

The fine difference between twisted and untwisted tubes in the spectral setting is reflected in the existence of (1.1) for the former.

In view of the spectral mapping theorem, the indifference (1.4) transfers to the following identity for the heat semigroup:

$$\forall t \geqslant 0, \quad \big\|e^{\Delta_D^{\Omega_\theta} t}\big\|_{L^2(\Omega_\theta)\to L^2(\Omega_\theta)} = e^{-E_1 t}, \tag{1.5}$$

irrespectively whether the tube Ω_θ is twisted or not. That is, we clearly have the exponential decay,

$$\big\|u(t)\big\|_{L^2(\Omega_\theta)} \leqslant e^{-E_1 t}\|u_0\|_{L^2(\Omega_\theta)}, \tag{1.6}$$

for each time $t \geqslant 0$ and any initial datum u_0 of (1.2). To obtain some finer differences as regards the time-decay of solutions, it is therefore natural to consider rather the *"shifted"* semigroup,

$$S(t) := e^{(\Delta_D^{\Omega} + E_1)t}, \tag{1.7}$$

as an operator from a *subspace* of $L^2(\Omega_\theta)$ to $L^2(\Omega_\theta)$.