Available online at www.sciencedirect.com
ScienceDirect
J. Math. Pures Appl. 89 (2008) 477–504
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Homogenization of nonlinear visco-elastic composites

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Received 10 November 2007
Available online 4 March 2008

Abstract

Quasi-static processes in nonlinear visco-elastic materials of solid-type are here represented by the system:

$$\sigma - B(x) : \frac{\partial \varepsilon}{\partial t} \in \beta(\varepsilon, x), \qquad -\operatorname{div} \sigma = \vec{f}, \tag{$*$}$$

coupled with initial and boundary conditions. Here σ denotes the stress tensor, ε the linearized strain tensor, $B(x)$ the viscosity tensor, $\beta(\cdot, x)$ a (possibly multi-valued) maximal monotone mapping, and $\vec{f}$ an applied load. Existence and uniqueness of the weak solution are proved.

A composite material in which the data β and B rapidly oscillate in space is then considered, and a two-scale model is derived via Nguetseng's notion of *two-scale convergence.* Although neither the stress nor the strain need be mesoscopically uniform, it is proved that their coarse-scale averages solve a global-in-time single-scale homogenized problem (*upscaling*). From any solution of the latter a solution of the two-scale problem is then reconstructed (*downscaling*). These results are at variance with the outcome of so-called *analogical models,* that assume a mean-field-type hypothesis. Finally, we represent the system ($*$) as a minimum problem, and interpret the above results in terms of two- and single-scale Γ-convergence.

Résumé

Des processus quasi-statiques pour des matériaux visco-élastiques de type solide sont décrits ici par le système :

$$\sigma - B(x) : \frac{\partial \varepsilon}{\partial t} \in \beta(\varepsilon, x), \qquad -\operatorname{div} \sigma = \vec{f}, \tag{$*$}$$

couplé à des conditions aux limites et initiales. Ici σ désigne le tenseur des contraintes, ε le tenseur des déformations linéarisées, $B(x)$ le tenseur de viscosité, $\beta(\cdot, x)$ un opérateur maximal monotone (éventuellement à valeurs multiples), et $\vec{f}$ une charge appliquée. L'existence et l'unicité de la solution faible sont établies.

Un matériau composite pour lequel β et B oscillent rapidement en espace est alors considéré, et un modèle à deux échelles est obtenu par la notion de *convergence à deux échelles* de Nguetseng. Bien que ni les contraintes ni les déformations n'ont besoin d'être uniformes à l'échelle mésoscopique, il est prouvé que leurs moyennes à l'échelle grossière sont solutions d'un problème homogénéisé à une échelle globale en temps (*upscaling*). Une solution du problème à deux échelles est alors reconstruite à partir d'une solution quelconque de ce dernier problème (*downscaling*). Ces résultats sont différents de ceux obtenus pour des *modèles dits analogiques,* qui supposent une hypothèse de type champ moyen. Finalement, nous représentons le système ($*$) sous la forme d'un problème de minimisation, et interprétons les résultats ci-dessus en termes de Γ-convergences à une ou deux échelles.

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0021-7824/$ – see front matter
doi:10.1016/j.matpur.2008.02.002

MSC: 35B27; 49J40; 74Q

Keywords: Homogenization; Two-scale convergence; Variational inequalities; Visco-elasticity

1. Introduction

In this paper we deal with the homogenization of processes in nonlinear visco-elastic composite materials via a two-length-scale homogenization procedure. First we couple the constitutive law with the quasi-static force-balance equation for a space-distributed system, and study an associated initial- and boundary-value problem. We then assume that the medium is a composite, derive two- and single-scale homogenized models, and prove their mutual equivalence.

Nonlinear visco-elasticity. We denote the region occupied by the body by Ω, the displacement with respect to the initial configuration by $\vec{u}$, the linearized strain-tensor by ε, the stress-tensor by σ, the density by ρ, and a distributed load by $\vec{f}$. Under the hypothesis of *infinitesimal* displacements, we represent a linearly viscous and nonlinearly elastic behavior via a constitutive relation of the form:

$$\sigma - B(x) : \frac{\partial \varepsilon}{\partial t} \in \beta(\varepsilon, x), \tag{1.1}$$

where $\beta(\cdot, x)$ is a (possibly multi-valued) maximal monotone mapping in the space of symmetric second-order tensors, and $B(x)$ is a positive-definite fourth-order tensor. This inclusion accounts for a solid-type viscous behavior, for ε may not indefinitely grow under constant stress whenever $\beta(\cdot, x)$ is surjective. This relation is tantamount to the variational inequality:

$$\left(\sigma - B(x) : \frac{\partial \varepsilon}{\partial t} - z\right) : (\varepsilon - v) \geqslant 0 \quad \forall (v, z) \text{ such that } z \in \beta(v, x). \tag{1.2}$$

The constitutive law (1.1) is not frame-independent, similarly to the classical linear theory that rests on the assumption of infinitesimal displacements. (1.1) might be regarded as an approximation of a finite-displacement relation. It would be in order to couple the constitutive relation (1.1) with the complete equation of continuum dynamics $\rho \partial^2 \vec{u}/\partial t^2 - \nabla \cdot \sigma = \vec{f}$ ($\nabla \cdot := \operatorname{div}$). However the time-scale of this equation is often shorter than that of (1.1), and we rather deal with the quasi-static force-balance equation:

$$-\nabla \cdot \sigma = \vec{f} \quad \text{in } \Omega \times]0, T[, \tag{1.3}$$

avoiding the analytical difficulties of quasilinear second-order hyperbolic equations. We couple the system (1.1) and (1.3) with appropriate initial- and boundary-conditions, provide a weak formulation in the framework of Sobolev spaces, and show existence and uniqueness of the solution. This part is based on classical techniques and is preparatory to the modeling of composites, which is the main concern of this work.

Analogical models. Large classes of rheological laws may be represented via so-called analogical models, that are constructed by arranging elementary univariate models in series and/or in parallel. In several cases the constitutive relations that are so derived may easily be carried over to the multivariate setting, although there the interpretation in terms of serial and parallel arrangements is meaningless. This technique is widely used in mechanics, in electromagnetism, in circuit theory, and so on, see e.g. [1,2,39,42,48,63]; however this procedure seems to be just heuristic, and in this paper we shall introduce an alternative approach.

Let us combine in series a finite family $\{M_j\colon\ j = 1, \ldots, N\}$ of models like (1.1), each one characterized by a nonlinear function β_j and a tensor function B_j. Once an initial condition has been specified for ε, for any j this defines a mapping $\mathcal{G}_j : \sigma_j \mapsto \varepsilon_j$ with memory. This arrangement has the following properties (omitting the variable t):

(i) (*discrete*) *mean-field hypothesis*: σ_j is independent of j and equals the stress σ of the overall model;
(ii) (*discrete*) *additivity*: $\varepsilon = \sum_{j=1}^{N} \varepsilon_j$, that is in our case

$$\varepsilon = \sum_{j=1}^{N} \mathcal{G}_j(\sigma_j) = \sum_{j=1}^{N} \mathcal{G}_j(\sigma) =: \hat{\mathcal{G}}(\sigma). \tag{1.4}$$