Applied Numerical Mathematics 112 (2017) 17–26

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Totally positive refinable functions with general dilation M

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ARTICLE INFO

Article history:
Received 14 November 2015
Received in revised form 16 July 2016
Accepted 6 October 2016
Available online 11 October 2016

Keywords:
Total positivity
Shape-preserving
Refinable function
General dilation
Subdivision scheme

ABSTRACT

We construct a new class of approximating functions that are M-refinable and provide shape preserving approximations. The refinable functions in the class are smooth, compactly supported, centrally symmetric and totally positive. Moreover, their refinable masks are associated with convergent subdivision schemes. The presence of one or more shape parameters gives a great flexibility in the applications. Some examples for dilation $M = 4$ and $M = 5$ are also given.

1. Introduction

Refinable functions with integer dilation $M \geq 2$ have a relevant role in several applications, like computer graphics and wavelets analysis as well as in the M-channel filter bank design, just to cite a few [4,7,17,18]. We emphasize also that *M-refinable functions* can be evaluated by suitable iterative schemes, namely *M-ary subdivision schemes*, which are strictly related to refinability.

While the case $M = 2$ has been thoroughly investigated, from the point of view of both the refinability and the subdivision (see, for instance, [4,7] and references therein), the case $M > 2$ did not receive as much attention, even if the use of a dilation greater than 2 allows one to achieve results not attainable in the binary case. For instance, the binary counterpart of the compactly supported symmetric orthonormal refinable functions with dilation $M \geq 3$ constructed in [5,12] does not exist. The same thing is true for the compactly supported interpolatory orthonormal refinable functions in [3]. Further properties of orthonormal refinable functions that apply just in the case when $M \geq 3$ can be found in [19]. Refinable functions with dilation $M = 3$ that are not orthonormal but are totally positive, were introduced in [11]. In this case the more flexibility given by dilation 3 is exploited to construct a wide class of symmetric refinable functions that have the same support and the same smoothness as the cardinal B-splines. Once again, the same result can not be achieved in the binary case; in fact, totally positive refinable functions with dilation $M = 2$ are less smooth than the cardinal B-splines having the same support [10].

Our aim is to construct M-refinable functions with general dilation $M > 2$ that provide shape preserving approximations. To this end, we will use the *total positivity* property of functions [13]. In fact, if a function f is totally positive, the *variation diminishing* property holds, i.e. for any sequence $\mathbf{g} = \{g(i), i \in I \subset \mathbb{Z}\}$ of finite support there results

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http://dx.doi.org/10.1016/j.apnum.2016.10.004
0168-9274/

$$S^-\Big(\sum_{i\in I} g(i)\, f(\cdot - i)\Big) \le S^-(\mathbf{g})\,, \tag{1.1}$$

where the symbol S^- denotes the strict sign changes of its argument. The variation diminishing property is stronger than other shape preserving properties, such as monotonicity or convexity preservation, since it implies that given a polygonal arc $\pi : A_0 A_1 \dots A_N$ with $A_i = (x_i, y_i) \in \mathbb{R}^2$, the curve $r(t) = \sum_{i=0}^N A_i\, f(t-i)$ *closely mimics* the shape of π [8]. It is then evident the interest that totally positive systems of functions take both in approximation theory and in the design of curves for CAGD applications.

In this paper we introduce a new class of M-refinable functions, with $M > 2$, and prove that they are totally positive. These functions share several properties, such as the compact support and the smoothness, with the *cardinal B-splines*, which are contained in it as a particular case. Nevertheless, these new refinable functions are more flexible in the applications because they depend on some shape parameters.

The paper is organized as follows. The definitions and some basic properties concerning refinable functions and subdivision schemes, are presented in Section 2. In Section 3, a class of palindromic polynomials is introduced and some properties of peculiar interest in this context are proved. Section 4 contains the main result of the paper, since here it is proved that any of the above polynomials gives rise to a totally positive refinable function. A detailed analysis of some particular refinable functions in the class having dilation $M = 4$ and $M = 5$ is presented in Section 5, where also the corresponding quaternary and 5-ary subdivision schemes are discussed. Moreover, we will construct a set of interpolatory quaternary subdivision schemes using a technique introduced in [1] to derive interpolatory schemes from approximating schemes with even arity. Even if these interpolatory schemes are not shape preserving, nevertheless we will show that they preserve monotonicity and convexity in same special cases.

2. Preliminaries

A *refinement equation* is a functional equation of the form

$$\varphi_{\mathbf{a}}(x) = \sum_{j\in\mathbb{Z}} a(j)\, \varphi_{\mathbf{a}}(M\,x - j)\,, \tag{2.1}$$

where M is the *dilation factor* and $\mathbf{a} = \{a(j),\ j \in \mathbb{Z}\}$ is the *refinable mask*. In the following, we assume that M is an integer ≥ 2, and $\mathbf{a}$ belongs to $l^0(\mathbb{Z})$, where $l^0(\mathbb{Z})$ denotes the linear space of finitely supported sequences on $\mathbb{Z}$.

Any solution φ of a refinement equation is called a *M-refinable function* and a necessary condition for its existence is

$$\sum_{j\in\mathbb{Z}} a(j) = M\,. \tag{2.2}$$

The best known example of M-refinable functions, for any dilation M, is provided by the cardinal B-splines of any degree n (for the expression of their masks, see, for instance, [11,14]).

To the mask $\mathbf{a}$ is associated the *symbol*, namely the Laurent polynomial given by its *z-transform*

$$P(z) = \sum_{j\in\mathbb{Z}} a(j)\, z^j\,, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{2.3}$$

Several properties of the refinable function φ can be deduced from the features of the associated mask. In particular, if $\mathbf{a}$ satisfies the *sum rules*

$$\sum_{i\in\mathbb{Z}} a(Mi + j) = 1\,, \qquad j = 0, 1, \dots, M-1, \tag{2.4}$$

then the refinement equation has at most one solution $\varphi \in L^1(\mathbb{R})$.

It is well known that refinability is strictly connected to *subdivision*. In fact, a given mask $\mathbf{a}$ gives rise to a *subdivision scheme* that, starting from an initial sequence $\mathbf{g}^0 = \{g^0(j),\ j \in \mathbb{Z}\}$, generates denser and denser sequences of points by the iterative rule

$$\mathbf{g}^{k+1} = S_{\mathbf{a}}\, \mathbf{g}^k\,, \qquad k \ge 0\,, \tag{2.5}$$

where $S_{\mathbf{a}}$ is the *subdivision operator* defined as

$$S_{\mathbf{a}}\, g(j) = \sum_{i\in\mathbb{Z}} a(j - Mi)\, g(i)\,, \qquad j \in \mathbb{Z}\,. \tag{2.6}$$

Under suitable conditions, the subdivision scheme (2.5) converges to a continuous function, i.e. there exists an uniformly continuous function $F_{\mathbf{g},\mathbf{a}}$ satisfying

$$\lim_{k\to\infty} \sup_{j\in\mathbb{Z}} |g^k(j) - F_{\mathbf{g},\mathbf{a}}(M^{-k} j)| = 0\,. \tag{2.7}$$