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A RBF-WENO finite volume method for hyperbolic conservation laws with the monotone polynomial interpolation method

Jingyang Guo, Jae-Hun Jung *

Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260-2900, USA

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ABSTRACT

Essentially non-oscillatory (ENO) and weighted ENO (WENO) methods are efficient high order numerical methods for solving hyperbolic conservation laws designed to reduce the Gibbs oscillations. The original ENO and WENO methods are based on the polynomial interpolation and the overall convergence rate is uniquely determined by the total number of interpolation points involved for the approximation. In this paper, we propose non-polynomial ENO and WENO finite volume methods in order to enhance the local accuracy and convergence. The infinitely smooth radial basis functions (RBFs) are adopted as a non-polynomial interpolation basis. Particularly we use the multi-quadratic and Gaussian RBFs. The non-polynomial interpolation such as the RBF interpolation offers the flexibility to control the local error by optimizing the free parameter. Then we show that the non-polynomial interpolation can be represented as a perturbation of the polynomial interpolation. To guarantee the essentially non-oscillatory property, the monotone polynomial interpolation method is introduced as a switching method to the polynomial reconstruction adaptively near the non-smooth area. The numerical results show that the developed non-polynomial ENO and WENO methods with the monotone polynomial interpolation method enhance the local accuracy and give sharper solution profile than the ENO/WENO methods based on the polynomial interpolation.

1. Introduction

Consider the hyperbolic conservation laws

$$v_t + \nabla \cdot F(v) = 0, \tag{1}$$

for the state vector $v \equiv v(t, x) : I \times \Omega \to \mathbb{R}^m$, where $I := (0, T]$ is a time interval with $T > 0$ and $\Omega \subset \mathbb{R}^d$ is an open bounded computational domain. $F(v) := [f_1(v), \cdots, f_m(v)]$ is the flux function. An initial condition $v_0(x) = v(x, 0)$ is given along with appropriate boundary conditions. Despite the smoothness of $v_0(x)$, the solution to (1) may develop a discontinuity within a finite time. High order numerical approximations of the developed discontinuity suffer from the Gibbs phenomenon yielding spurious oscillations near the discontinuity. Since the publications by Harten et al. [13] and by Jiang and Shu [15], the essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) methods have been one of the

* Corresponding author.
E-mail addresses: jguo4@buffalo.edu (J. Guo), jaehun@buffalo.edu (J.-H. Jung).

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Table 1
Commonly used radial basis functions $\phi(r)$, $r \geqslant 0$ with ϵ known as the shape parameter.

Infinitely smooth RBFs		Piecewise smooth RBFs	
Gaussian (GA)	$\exp(-(\epsilon r)^2)$	Polyharmonic spline	r^k, $k=1,3,5,\ldots$
Multiquadratic (MQ)	$\sqrt{1+(\epsilon r)^2}$		$r^k \ln(r)$, $k=2,4,6,\ldots$
Inverse quadratic (IQ)	$\frac{1}{1+(\epsilon r)^2}$		

most powerful numerical methods that can successfully deal with the Gibbs oscillations. Numerous modifications of the original ENO/WENO methods have been also developed, while resolving small scale structures accurately and efficiently. These include recent works such as WENO-M [14], WENO-Z [3], power-ENO [20], WENO-P [11], modification of the ENO basis [2] and WENO-η [8] methods, to name a few. There is no best ENO/WENO variation because all variations have their own strengths and weaknesses. However, most variations have a common ground: the polynomial reconstruction. In recent reviews of the WENO method by Shu [23], the WENO reconstruction based on non-polynomial functions such as the Fourier functions is briefly mentioned [5].

In this paper, we present a simple new type of the ENO/WENO methods based on non-polynomial interpolations. As an example of non-polynomial bases, radial basis functions (RBFs) are used. In [1], the ADER method was developed based on the polyharmonic spline, which belongs to the family of piecewise smooth RBFs. The motivation of the method presented in [1] was to adopt the WENO method efficiently for the arbitrary geometry and unstructured mesh by using the meshless feature of RBFs. So there was no undetermined shape parameter — or the shape parameter is fixed as $\epsilon = 1$. The order of convergence is overall fixed once the size of each stencil k is fixed. Our main motivation in this paper, however, is to enhance the original ENO/WENO accuracy by modifying the interpolation coefficients. For this reason, we need free parameters to optimize, which makes the presented method in this paper different from the one in [1].

RBFs are divided into two categories depending on whether there are undetermined shape parameters: piecewise smooth RBFs and infinitely smooth RBFs (see Table 1). In this paper, we first use the infinitely smooth RBFs because they are defined with a free parameter ϵ, so-called the shape parameter. Since the parameter is free yet to be determined locally, it yields the flexibility to improve the original ENO/WENO accuracy. In fact, different RBFs give equivalent interpolations. This means we will end up with the same type of reconstruction irrespective of the bases used. This is also true for the piecewise smooth RBF basis used in [1], if we regard them as a special case of the infinitely smooth RBF basis with the shape parameter fixed as $\epsilon = 1$. We can also show that the derived RBF interpolation formulas are equivalent to the perturbed polynomial interpolation. Thus one can use other non-polynomial bases rather than RBFs as long as the new basis is defined with one or more free parameters for improving the local accuracy and convergence. For the RBF interpolation, it becomes a polynomial interpolation if the shape parameter vanishes. This makes it easy to modify the existing ENO/WENO code to the proposed ENO/WENO methods. We restrict our discussion to the one-parameter perturbation although it may be possible to utilize multiple free parameters.

Unlike the polynomial interpolation, the perturbed polynomial interpolation such as the RBF interpolation is not necessarily consistent, i.e. reconstruction coefficients may not sum to unity. Such an inconsistency helps the proposed method to enhance local accuracy in the smooth area. However, if the solution contains discontinuities, the inconsistent reconstruction causes the Gibbs oscillations. To prevent the Gibbs oscillations, the monotone polynomial interpolation method by measuring the local extrema is introduced. The non-polynomial interpolation is switched into the polynomial interpolation in the non-smooth region. This can be done easily by adopting the vanishing shape parameter to reduce the method into the polynomial method [16].

The paper is composed of the following sections. In Section 2, we briefly explain the finite volume ENO/WENO methods. In Section 3, we use the case of $k = 2$ to illustrate the RBF-ENO interpolation based on the multi-quadric (MQ), Gaussian RBFs and the perturbed polynomial. In this section, the tables of the reconstruction coefficients for $k = 2$ and $k = 3$ are provided. In Section 4, we explain the monotone polynomial interpolation method in detail. In Section 5, we briefly explain the time-integration and flux schemes that are used for the numerical experiment. Then the 1D numerical examples are presented for both scalar and system problems. In Section 6, we explain the 2D ENO/WENO finite volume interpolation method based on the non-polynomial bases. In Section 7 the 2D numerical examples are presented. In Section 8, we provide a brief conclusion and our future research.

2. Finite volume ENO/WENO method

Suppose that we are given a grid with N cells such that

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = b.$$

For the i-th cell $I_i = [x_{i-\frac{1}{2}},\ x_{i+\frac{1}{2}}]$, define the cell center x_i and cell size Δx_i as

$$x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}),\ \Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}},\ i = 1, 2, \cdots, N,$$