Applied Numerical Mathematics 112 (2017) 79–90

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A spectral iterative method for solving nonlinear singular Volterra integral equations of Abel type

A. Shoja [a,*], A.R. Vahidi [b], E. Babolian [a]

[a] Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
[b] Department of Mathematics, Yadegar-E-emam khomeni Branch, Islamic Azad University, Tehran, Iran

A R T I C L E I N F O

Article history:
Received 19 March 2016
Received in revised form 4 September 2016
Accepted 6 September 2016
Available online xxxx

Keywords:
Iterative method
Spectral method
Fractional differential equation
Volterra integral equation
Collocation method

A B S T R A C T

In this paper, a spectral iterative method is employed to obtain approximate solutions of singular nonlinear Volterra integral equations, called Abel type of Volterra integral equations. The Abel's type nonlinear Volterra integral equations are reduced to nonlinear fractional differential equations. This approach is based on a combination of two different methods, i.e. the iterative method proposed in [7] and the spectral method. The method reduces the fractional differential equations to systems of linear algebraic equations and then the resulting systems are solved by a numerical method. Finally, we prove that the spectral iterative method (SIM) is convergent. Numerical results comparing this iterative approach with alternative approaches offered in [4,8,24] are presented. Error estimation also corroborate numerically.

1. Introduction

Many problems of mathematical physics can be formulated in the form of integral equations. These equations can also be reformulated in other mathematical problems such as partial differential equations and ordinary differential equations. Recently, there has been a growing interest in the Volterra integral equations in many fields of physics and engineering [3], for example, potential theory and Dirichlet problems, electrostatics, the particle transport problems of astrophysics and reactor theory, contact problems, diffusion problems, and heat transfer problems has arisen.

Many researchers have applied some valid numerical methods to solve integral equations using various polynomials [25]. Recently, Maleknejad et al. [20] and Tari and Shahmorad [29] applied computational method to solve two-dimensional linear Fredholm integral equations of the second kind. Shahsavaran [26] has used block pulse functions and Taylor Expansion method. Bellour and Rawashdeh [5] and Wang [30] also used Taylor polynomials with computer algebra. Bhatti and Bracken [6] used Bernstein polynomials to solve second order linear and first order nonlinear differential equations. Shirin and Islam [28], also have applied polynomials for solving Fredholm integral equations of the second kind. An augmented Galerkin technique was described by Amaratunga [2] for the numerical solution of one-dimension partial differential equation. Hanna and Kucera [13] also studied collocation to solve Fredholm integral equations. In [16] a solution of integral equation via Laguerre polynomials is suggested. A numerical solution of the integral equations of the first kind is studied by Rashed [23].

One of the integral equations that arises from physical or mechanical models without passing through a differential equation is the one by Abel [1]. In many branches of science such as astronomy, quantum mechanics, optics such singular

* Corresponding author.
E-mail address: Ahmadshoja1@yahoo.com (A. Shoja).

http://dx.doi.org/10.1016/j.apnum.2016.09.008
0168-9274/© 2016 Published by Elsevier B.V. on behalf of IMACS.

Volterra integral equations arise. We have presented in [9,10] numerical methods for elastostatic problems, solving systems of singular integral equations. A detailed explanation of singular integral equation with Cauchy and Dirichlet problems is presented in [19]. The linear and nonlinear integral equations are solved in [31]. Li Huang et al. [15] have presented a stable approximate inversion of Abel integral equation by using the Taylor expansion. Zeilon [32] studied the generalized type of the Abel integral equation upon a finite segment.

The nonlinear Abel type Volterra integral equation was studied by Diogo and Lima and Rebelo [8], where they explain the extrapolation method to solve equation and then compare it with the collocation methods and graded meshes. The nonlinear Abel equation was solved by P. Kumar Gupt [12] using modified Adomian decomposition method. The fractional differential equations were solved by Kazem et al. [17] using operational matrices of fractional-order Legendre functions.

In this paper, we mix together collocation spectral method and an iterative procedure to solve nonlinear Abel type Volterra integral equations. Then we apply this method to some problems and show its good accuracy. In this method, the nonlinear Abel type Volterra integral equation is reduced to nonlinear fractional differential equation and the solution is done through a combination of Legendre spectral method of fractional-order with the iterative method proposed in [7]. This paper is organized as follows.

We describe preliminaries in Sec. 2. In Sec. 3 we describe how an Abel's type Volterra integral equation reduces to a fractional differential equation. In Sec. 4.1 we describe the iterative method and in Sec. 4.2 we give a description of shifted fractional-order Legendre functions. In Sec. 4.3 we use collocation method to obtain the approximate solution for differential equation with initial conditions as a linear combination of fractional-order Legendre functions. In Sec. 4.4, we describe the spectral-iterative method (SIM) which is a combination of two different methods, one iterative and the other spectral. We prove the convergence of SIM in Sec. 4.5 and introduce the numerical results in Sec. 5 and investigate the estimation of errors in Sec. 6.

2. Preliminaries and notations

In order to proceed, we need the following definitions of fractional derivatives and integrals. First, we introduce the Caputo definition of fractional integral operator J_a^{α}.

Definition 2.1. Let $\alpha \in R^+$. The operator J_a^{α}, defined on the space $L_1[a,b]$ by

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \tag{1}$$

$$J_a^0 f(x) = f(x),$$

for $a \le x \le b$, is called the Caputo fractional integral operator of order α.

Properties of the operator J_a^{α} can be found in [22]. For $f \in L_1[a,b]$, $\alpha, \beta \ge 0$ and $\gamma > -1$, we mention only the following:

(1) $J_a^{\alpha} f(x)$ *exists for almost every* $x \in [a,b]$,
(2) $J_a^{\alpha} J_a^{\beta} f(x) = J_a^{\alpha+\beta} f(x)$,
(3) $J_a^{\alpha} J_a^{\beta} f(x) = J_a^{\beta} J_a^{\alpha} f(x)$,
(4) $J_a^{\alpha} (x-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} (x-a)^{\alpha+\gamma}$.

Definition 2.2. The fractional derivative of f in the Caputo sense is defined as

$$D_a^{\alpha} f(x) = J_a^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} \frac{d^m}{dt^m} f(t)dt, \tag{2}$$

where $m \in N$, $m-1 < \alpha \le m$, and $f \in L_1[a,b]$.

Properties of the operator D_a^{α} can be found in [18,21,22]. For $m-1 < \alpha \le m$, $x > a$ and $\gamma > -1$ we mention only the following:

(1) $D_a^{\alpha} (x-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (x-a)^{\gamma-\alpha}$,
(2) $D_a^{\alpha} J_a^{\alpha} f(x) = f(x)$.