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Adaptive multistep time discretization and linearization based on a posteriori error estimates for the Richards equation

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ABSTRACT

We derive some a posteriori error estimates for the Richards equation. This parabolic equation is nonlinear in space and in time, thus its resolution requires fixed-point iterations within each time step. We measure the approximation error with the dual norm of the residual. A computable upper bound of this error consists of several estimators involving adequate reconstructions based on the degrees of freedom of the scheme. The space and time reconstructions are specified for a two-step backward differentiation formula and a discrete duality finite volume scheme. Our strategy to decrease the computational cost relies on an aggregation of the estimators in three components: space discretization, time discretization, and linearization. We propose an algorithm to stop the fixed-point iterations after the linearization error becomes negligible, and to choose the time step in order to balance the time and space errors. We analyze the influence of the parameters of this algorithm on three test cases and quantify the gain obtained in comparison with a classical simulation.

1. Introduction

Prediction of fluid flows in porous media is a challenging problem due to the space heterogeneity, the nonlinearity of the hydrodynamic laws, and the anisotropy of the conductivity. In this paper, we study an adaptive method, based on a posteriori error estimation, in order to optimize the computational cost of our simulations. We focus on water infiltration modeled by the Richards equation, written here on a polygonal domain Ω (in $\mathbb{R}^2$) with a finite time horizon $T > 0$ and mixed Dirichlet–Neumann conditions on the boundary $\partial\Omega = \partial\Omega^{\mathrm{D}} \sqcup \partial\Omega^{\mathrm{N}}$,

$$\begin{cases} \partial_t \theta(\psi) - \nabla \cdot (\mathbb{K}(\psi)\nabla(\psi + z)) = f & \text{in } Q_T := \Omega \times (0,T), \\ \psi = \psi_{\mathrm{D}} & \text{on } \partial\Omega^{\mathrm{D}} \times (0,T), \\ -\mathbb{K}(\psi)\nabla(\psi + z) \cdot n_\Omega = \mathbf{v}_{\mathrm{N}} & \text{on } \partial\Omega^{\mathrm{N}} \times (0,T), \\ \psi_{t=0} = \psi^0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{1}$$

where the unknown ψ is the pressure head, ψ_{D} and $\mathbf{v}_{\mathrm{N}}$ are respectively the Dirichlet and Neumann prescribed data, z is the vertical coordinate and n_Ω is the outward unit normal to the domain. The water content $\psi \mapsto \theta(\psi)$ and the

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conductivity $\psi \mapsto \mathbb{K}(\psi)$ are two constitutive laws closing the model (see, e.g. [24,36]). The water content is assumed to be a non-decreasing, Lipschitz continuous function, while the conductivity takes the form $\mathbb{K}(\psi) = \overline{\mathbb{K}}(x)k(\psi)$, where $\overline{\mathbb{K}}(x)$ is a two-dimensional matrix with real coefficients. Here, k is a continuous, bounded function with $0 < k_{\min} < k(\psi) < k_{\max}$, and $\overline{\mathbb{K}}$ is a bounded measurable function with uniform ellipticity. A various choice of methods is available to approximate the Richards equation (see [7] for instance) and we choose a Discrete Duality Finite Volume (DDFV) scheme because it has proven robustness for anisotropic heterogeneous diffusion problems on general, most notably non-matching and distorted meshes. Besides it preserves the symmetry of the continuous problem, offers a L^2-norm superconvergence on various examples, together with a very accurate approximation of the gradient, as was noted by Herbin et al. in [25]. In this paper we consider a version of the DDFV scheme proposed by Coudière et al. [13] and Krell [26] allowing discontinuities of the permeability tensor along faces. Concerning time discretization, implicit schemes are preferred to solve this nonlinear parabolic equation, since explicit ones are only valid in the vadose zone and have a restrictive CFL condition due to the diffusion term. We propose to use the second-order backward differentiation formula (BDF2), rather than the Crank–Nicolson scheme which is not unconditionally L^∞-stable [23]. Moreover, the BDF schemes are more efficient than the implicit Runge–Kutta ones which are multistage and ill-posed if the soil is saturated, as was underlined in [35]. Therefore, the resolution of (1) requires to solve nonlinear systems at each discrete time, and we have to pay special attention to the computational cost.

In order to optimize this cost, we propose to use local a posteriori estimates, which allow to control the error between the exact and approximate solutions. These estimates are local bounds that involve only the approximate solution and, ideally, some fully computable constants. Several methods are available to obtain such estimates. The fashionable *explicit residual method*, introduced by Babuška and Rheinboldt [4] and extended by Verfürth [37], bounds the error in the energy norm. It uses an evaluation of local residuals with respect to the strong form of the equation, but most often involves a multiplicative constant hard to compute [38]. For the linear heat equation, an upper bound of the error is obtained by Picasso [30] in the energy norm and by Makridakis and Nochetto [29] in higher-order norms. The elliptic reconstruction developed in [28,29] is particularly important to derive error estimators in the presence of mesh adaptation [6]. The *Zienkiewicz–Zhu estimator* [39] estimates locally the gradient of the exact solution through the approximate solution. Although this method is popular thanks to its simplicity and low computational cost, a guaranteed estimate is not provided. The *hierarchical method* enhances the approximation space (by h and p-refinement) to compute the estimators. The *equilibrated residual method* solves local boundary value problems where the boundary data are chosen to equilibrate the interior residual. The paper of Bank and Weiser [5] contains fundamental ideas (most notably the saturation assumption and the equilibration of boundary data) endorsed by these two methods. All these techniques are reviewed by Ainsworth and Oden in [2] for finite element methods. In the recent paper [16], a new residual error is developed for a BDF-Discontinuous Galerkin (DG) method applied to compressible flows. However it lacks some theoretical background to ensure that the computed error is directly linked to the actual error of the solution. Regarding finite volume methods, Ohlberger gets a L^1-estimate for nonlinear convection–diffusion equations, respectively for cell-centered [31] and vertex-centered [32] schemes. An adaptive mesh refinement strategy is proposed, as in [8] where a mesh adaptation is performed for the Richards equation with mixed finite elements. Estep et al. [20] derive a posteriori error estimates for a convection–diffusion–reaction equation discretized by a cell-centered finite volume scheme. They solve an adjoint problem to obtain an equivalence with a mixed finite element method. More specifically, for a DDFV scheme, Omnes [33] derives an estimate in the energy norm from a variational formulation which allows to use standard finite element techniques. This estimate is written both on a primary and a secondary mesh, and calls on an Helmholtz–Hodge decomposition.

The current work deals with the *equilibrated fluxes method* [34] based on the reconstruction of some continuous equilibrated fluxes from the discrete ones. This method has received particular attention in various studies over the last few years: finite elements for elasticity problems and the Poisson equation in [14,27], DG methods for a convection–reaction–diffusion equation in [18], finite volumes for multiphase compositional flows in [15]. More recently, theoretical developments have unified various space discretizations, for the linear heat equation [19] and for a nonlinear parabolic problem [17]. In the latter, robust fully computable lower and upper bounds were obtained using a space–time dual norm. This work includes the water content formulation of the Richards equation. Other results concerning the Richards equation using the Kirchhoff transform are available in [9]. We discard the Kirchhoff transform to benefit from conservative schemes especially designed for physical variables.

In this paper we consider a formulation based on the pressure head, which remains valid in saturated soils unlike the water content form. Our objective is to balance the space and time discretization errors after neglecting the effect of linearizations. More specifically, we work on a fixed mesh and adapt the time step and the number of nonlinear iterations. To this aim, we derive a fully computable upper bound in the spirit of [17,34], while a lower bound for any finite volume scheme requires more theoretical developments out of the objective of this work. We propose some space and time flux reconstructions for the BDF2-DDFV scheme. Indeed, the nonlinearity in the time derivative term requires a special treatment, which we address by equilibrating the time flux as well as the space flux in accordance with the space–time norm used in the estimators. Then the upper bound can be split into spatial, temporal and linearization error components. An adaptive algorithm is proposed to choose a stopping criterion for the nonlinear algorithm and to adjust the time step during the simulation. Although we consider a DDFV scheme in this paper, our results remain general and not attached to a specific discretization.

The outline is as follows. Section 2 introduces the continuous and discrete frameworks. Section 3 specifies our upper bounds using space and time equilibrated flux assumptions. Section 4 reformulates the BDF2 scheme as a one-step scheme