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Convergence of Newton, Halley and Chebyshev iterative methods as methods for simultaneous determination of multiple polynomial zeros

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ABSTRACT

In this paper, we provide a local convergence analysis of Newton, Halley and Chebyshev iterative methods considered as methods for simultaneous determination of all multiple zeros of a polynomial f over an arbitrary normed field $\mathbb{K}$. Convergence theorems with a priori and a posteriori error estimates for each of the proposed methods are established. The obtained results for Newton and Chebyshev methods are new even in the case of simple zeros. Three numerical examples are given to compare the convergence properties of the considered methods and to confirm the theoretical results.

1. Introduction

There is no doubt that the most popular iterative methods in the literature are Newton's method, Halley's method [6] and Chebyshev's method [2]. For a detailed historical survey of these famous iterative methods we refer the reader to the elegant papers of Ypma [35], Scavo and Thoo [28] and Ezquerro et al. [3].

It is well known that Newton's method converges quadratically while Halley and Chebyshev methods converge cubically to simple zeros, but all of these methods converge only linear for multiple zeros. In 1870, Schröder [29] has presented a modification of Newton's method which restores the quadratic convergence in the case of zero with known multiplicity m. In 1963, Obreshkov [12] has developed modifications of Halley and Chebyshev methods which converge cubically to a zero with multiplicity m. The best results for these three modifications applied to polynomials are due to Proinov [16], Proinov and Ivanov [23] and Ivanov [8], respectively. Recall that, to preserve the order of convergence, Traub [33] has suggested to use any method for f^{m-1} or $f^{1/m}$ instead of f.

Throughout this paper $(\mathbb{K}, |\cdot|)$ denotes an arbitrary normed field and $\mathbb{K}[z]$ denotes the ring of polynomials over $\mathbb{K}$. Besides, I_n denotes the set of indices $1, \dots, n$, *i.e.*, $I_n = \{1, \dots, n\}$. Furthermore, the vector space $\mathbb{K}^n$ is equipped with the cone norm $\|x\| = (|x_1|, \dots, |x_n|)$ with values in $\mathbb{R}^n$, maximum norm defined by $\|x\|_\infty = \max_{i \in I_n} |x_i|$ and $\preceq$ denotes the coordinatewise ordering in $\mathbb{R}^n$.

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Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. There are a lot of iterative methods for finding all zeros of f simultaneously (see, *e.g.*, the monographs of Sendov et al. [32], Kyurkchiev [10] and Petković [13]). The most famous among these methods is due to Weierstrass [34] in 1891.

In 2002, Batra [1] has established a semilocal convergence theorem for Newton's method considered as a method for finding all zeros of a complex polynomial f simultaneously. He introduced in $\mathbb{C}^n$ the following iteration

$$x^{k+1} = x^k - N_f(x^k), \qquad k = 0, 1, 2, \ldots,$$

where the *Newton's operator* N_f is defined in $\mathbb{C}^n$ by

$$N_f(x) = (N_1(x), \ldots, N_n(x)) \quad \text{with} \quad N_i(x) = \frac{f(x_i)}{f'(x_i)}.$$

In 2015, Proinov and Ivanov [24] have proved two types of local convergence theorems as well as a semilocal convergence theorem for Halley's method as a method for simultaneous computation of all simple zeros of a polynomial f over an arbitrary normed field $\mathbb{K}$. For this purpose, they have defined the Halley's iteration in $\mathbb{K}^n$ as follows

$$x^{k+1} = x^k - H_f(x^k),$$

where *Halley's operator* H_f is defined in $\mathbb{K}^n$ by $H_f(x) = (H_1(x), \ldots, H_n(x))$ with

$$H_i(x) = \frac{f(x_i)}{f'(x_i)} \left(1 - \frac{1}{2} \frac{f(x_i)}{f'(x_i)} \frac{f''(x_i)}{f'(x_i)}\right)^{-1}.$$

On the other hand, iterative methods for the simultaneous computation of multiple polynomial zeros, have been investigated rarely in the literature. These methods most frequently have been developed as modifications of the known methods for simple zeros. Nevertheless, after 1972 when Sekuloski [30] has presented the first such method, many scientific works have been devoted to this topic (see, *e.g.*, Farmer and Loizou [4], Gargantini [5], Semerdziev [31], Petković et al. [15], Kjurkchiev and Andreev [9], Iliev and Iliev [7], Proinov and Cholakov [22] and the references therein). Motivated by the mentioned works, as a continuation of [1] and [24], in this paper we study Newton, Halley and Chebyshev methods as methods for simultaneous computation of all multiple zeros of a polynomial f over an arbitrary normed field $\mathbb{K}$.

Let $\xi_1, \ldots, \xi_s$ be all distinct zeros of f with multiplicities $m_1, \ldots, m_s$, respectively ($s \geq 2$). We define Newton, Halley and Chebyshev iterations in $\mathbb{K}^s \subset \mathbb{K}^n$ by

$$x^{k+1} = N_f(x^k), \quad x^{k+1} = H_f(x^k) \quad \text{and} \quad x^{k+1} = C_f(x^k), \tag{1.1}$$

where *Newton's operator* N_f is defined in $\mathbb{K}^s$ by $N_f(x) = (N_1(x), \ldots, N_s(x))$ with

$$N_i(x) = \begin{cases} x_i - m_i \dfrac{f(x_i)}{f'(x_i)} & \text{if } f'(x_i) \neq 0, \\ x_i & \text{if } f'(x_i) = 0, \end{cases} \tag{1.2}$$

Halley's operator H_f is defined in $\mathbb{K}^s$ by $H_f(x) = (H_1(x), \ldots, H_s(x))$ with

$$H_i(x) = \begin{cases} x_i - \dfrac{f(x_i)}{f'(x_i)} \left(\dfrac{m_i + 1}{2m_i} - \dfrac{1}{2} \dfrac{f(x_i)}{f'(x_i)} \dfrac{f''(x_i)}{f'(x_i)}\right)^{-1} & \text{if } f'(x_i) \neq 0, \\ x_i & \text{if } f'(x_i) = 0 \end{cases} \tag{1.3}$$

and *Chebyshev's operator* C_f is defined in $\mathbb{K}^s$ by $C_f(x) = (C_1(x), \ldots, C_s(x))$ with

$$C_i(x) = \begin{cases} x_i - m_i \dfrac{f(x_i)}{f'(x_i)} \left(\dfrac{3 - m_i}{2} + \dfrac{m_i}{2} \dfrac{f(x_i)}{f'(x_i)} \dfrac{f''(x_i)}{f'(x_i)}\right) & \text{if } f'(x_i) \neq 0, \\ x_i & \text{if } f'(x_i) = 0. \end{cases} \tag{1.4}$$

The main aim of this paper is to establish local convergence theorems with a priori and a posteriori error estimates for Newton, Halley and Chebyshev iterations (1.1). We shall note that the obtained results for Newton and Chebyshev methods are new even in the case of simple zeros.

2. Convergence analysis

Recently, Proinov [17–21] has presented general convergence theorems for the Picard iteration

$$x_{k+1} = Tx_k, \tag{2.1}$$

where $T: D \subset X \to X$ is an iteration function in a cone metric space (X, d) over a solid vector space $(Y, \preceq)$. Throughout this paper, we follow the terminology from [17–19]. For the proof of our main results, we apply the following theorem of Proinov [19].