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The computation of the degree of an approximate greatest common divisor of two Bernstein polynomials

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A B S T R A C T

This paper considers the computation of the degree t of an approximate greatest common divisor $d(y)$ of two Bernstein polynomials $f(y)$ and $g(y)$, which are of degrees m and n respectively. The value of t is computed from the QR decomposition of the Sylvester resultant matrix $S(f,g)$ and its subresultant matrices $S_k(f,g)$, $k=2,\dots,\min(m,n)$, where $S_1(f,g)=S(f,g)$. It is shown that the computation of t is significantly more complicated than its equivalent for two power basis polynomials because (a) $S_k(f,g)$ can be written in several forms that differ in the complexity of the computation of their entries, (b) different forms of $S_k(f,g)$ may yield different values of t, and (c) the binomial terms in the entries of $S_k(f,g)$ may cause the ratio of its entry of maximum magnitude to its entry of minimum magnitude to be large, which may lead to numerical problems. It is shown that the QR decomposition and singular value decomposition (SVD) of the Sylvester matrix and its subresultant matrices yield better results than the SVD of the Bézout matrix, and that $f(y)$ and $g(y)$ must be processed before computations are performed on these resultant and subresultant matrices in order to obtain good results.

1. Introduction

The computation of the greatest common divisor (GCD) of two polynomials occurs in several applications, including image processing, control systems, robotics and the computation of intersections of Bézier curves and surfaces in computer aided geometric design [4]. The GCD is defined for exact polynomials only, but practical problems yield inexact polynomials because their coefficients are corrupted by noise. It is therefore necessary to consider an approximate greatest common divisor (AGCD) of noisy forms $f(y)$ and $g(y)$ of, respectively, the exact polynomials $\hat{f}(y)$ and $\hat{g}(y)$. The GCD of $\hat{f}(y)$ and $\hat{g}(y)$ is unique up to an arbitrary non-zero constant, but an AGCD of $f(y)$ and $g(y)$ is not unique because it can be defined in several ways. Furthermore, each AGCD may be considered to be the GCD of polynomials that lie in neighbourhoods of $f(y)$ and $g(y)$, and these AGCDs may not be unique, apart from scaling.

An AGCD of two polynomials and methods for its computation are discussed in Section 2, and it is shown in Section 3 that the degree of the GCD of $\hat{f}(y)$ and $\hat{g}(y)$, which are of degrees m and n respectively, can be calculated from the rank of their Sylvester matrix $S(\hat{f},\hat{g})$ and its subresultant matrices $S_k(\hat{f},\hat{g})$, $k=2,\dots,\min(m,n)$, where $S_1(\hat{f},\hat{g})=S(\hat{f},\hat{g})$.

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Consideration of the entries of these matrices shows it is convenient to rearrange them in order to reduce the computational complexity of their evaluation. This rearrangement leads to Section 4, where an equation that allows $S_k(\hat{f}, \hat{g})$ to be computed from $S_j(\hat{f}, \hat{g})$, $j < k$, is developed.

It is shown in Section 5 that $S_k(\hat{f}, \hat{g})$, $k = 1, \ldots, \min(m, n)$, must be processed by three operations before computations are performed on these matrices in order to minimise numerical problems that may arise. Methods for the computation of the degree of an AGCD of $f(y)$ and $g(y)$ are discussed in Section 6, and Section 7 contains examples of this computation. The contents of the paper are summarised in Section 8.

2. An approximate greatest common divisor

The following definition of an AGCD of two polynomials is used by Bini and Boito [2]. It involves concepts of the nearness of two polynomials, the maximum degree of a polynomial from the set of polynomials that satisfy the nearness condition, and a measure of the distance between two polynomials.

Definition 2.1. Let $f(y)$ and $g(y)$ be polynomials of degrees m and n respectively. A polynomial $d(y)$ is an ϵ-divisor of $f(y)$ and $g(y)$ if there exist polynomials $\tilde{f}(y)$ and $\tilde{g}(y)$, of degrees m and n respectively, such that

$$\left\| f(y) - \tilde{f}(y) \right\| \leq \epsilon \left\| f(y) \right\| \qquad \text{and} \qquad \left\| g(y) - \tilde{g}(y) \right\| \leq \epsilon \left\| g(y) \right\|,$$

and $d(y)$ divides $\tilde{f}(y)$ and $\tilde{g}(y)$. If $d(y)$ is an ϵ-divisor, of maximum degree, of $f(y)$ and $g(y)$, then it is called an ϵ-GCD, or AGCD, of $f(y)$ and $g(y)$. The polynomials $u(y) = \tilde{f}(y)/d(y)$ and $v(y) = \tilde{g}(y)/d(y)$ are called ϵ-cofactors.

This definition of an AGCD of $f(y)$ and $g(y)$ is a function of ϵ, the maximum value of the upper bound of the relative error between $f(y)$ and $\tilde{f}(y)$, and $g(y)$ and $\tilde{g}(y)$. The value of ϵ may not be known, or it may only be known approximately, in which case this definition of an AGCD may not be appropriate. Another definition, which uses subresultant matrices of the Sylvester matrix of $f(y)$ and $g(y)$, is therefore considered in Section 3.

Previous work on the computation of an AGCD of two power basis polynomials has used the QR decomposition [7,20] and the singular value decomposition (SVD) [6,9] of the Sylvester matrix. Also, optimisation methods [5,21] and methods that exploit the structure of the Sylvester matrix [1,2,11,12,21] have been used. The methods described in these papers require that the threshold ϵ be specified, and common divisors of degree k, $k = \min(m, n)$, $\min(m, n) - 1$, $\min(m, n) - 2, \ldots, 2, 1$, are computed and an error measure is calculated for each value of k. The procedure terminates at the first (largest) value of k for which the error measure is less than ϵ.

It was noted above that ϵ may not be known in practical problems, or it may only be known approximately. Previous work has shown, however, that if $f(y)$ and $g(y)$ are preprocessed, then computations on the Sylvester matrix and its subresultant matrices enable the degree t of an AGCD to be computed, even when the value of ϵ, or bounds on its value, are not known [17]. This method for the determination of t has been used for the computation of a structured low rank approximation of the Sylvester matrix [16] and multiple roots of a polynomial [14,18].

The computation of an AGCD of two Bernstein polynomials is considered in [3,19], and the work described in this paper extends the work in these two papers. The application of Euclid's algorithm to the computation of the GCD of two Bernstein polynomials is considered in [13], but unsatisfactory results are obtained and the need for robust methods for this computation is emphasised.

3. The Sylvester matrix and the degree of the GCD

This section considers the calculation of the degree of the GCD of $\hat{f}(y)$ and $\hat{g}(y)$, and it is shown that it reduces to the computation of the rank of each matrix $S_k(\hat{f}, \hat{g})$, $k = 1, \ldots, \min(m, n)$. The discussion in this section is brief, and more details are in [19].

If the degree of the GCD of $\hat{f}(y)$ and $\hat{g}(y)$ is $\hat{t}$, then, for each value of $k = 1, \ldots, \hat{t} - 1$, $\hat{f}(y)$ and $\hat{g}(y)$ have more than one common divisor of degree k, and they have only one common divisor, to within an arbitrary non-zero scalar multiplier, of degree $\hat{t}$. It follows that if $\hat{d}_k(y)$ is a common divisor of degree k, there exist quotient polynomials $\hat{u}_k(y)$ and $\hat{v}_k(y)$, which are of degrees $m - k$ and $n - k$ respectively, such that

$$\hat{f}(y) = \hat{u}_k(y)\hat{d}_k(y) \qquad \text{and} \qquad \hat{g}(y) = \hat{v}_k(y)\hat{d}_k(y).$$

It is shown in [19] that these equations can be expressed in matrix form as

$$D_k^{-1} T_k(\hat{f}, \hat{g}) Q_k \begin{bmatrix} \hat{v}_k \\ -\hat{u}_k \end{bmatrix} = 0, \qquad k = 1, \ldots, \min(m, n), \tag{1}$$

where $D_k^{-1} \in \mathbb{R}^{(m+n-k+1)\times(m+n-k+1)}$ is given by