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Discontinuous Galerkin methods with interior penalties on graded meshes for 2D singularly perturbed convection–diffusion problems

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A B S T R A C T

In this paper, we introduce discontinuous Galerkin methods with interior penalties, both the NIPG and SIPG method for solving 2D singularly perturbed convection–diffusion problems. On the modified graded meshes with the standard Lagrange $\mathcal{Q}_k$-elements ($k = 1, 2$), we show optimal order error estimates in the ε-weighted energy norm uniformly, up to a logarithmic factor, in the singular perturbation parameter ε. We prove that the convergence rate in the ε-weighted energy norm is $\mathcal{O}\left(\frac{\log^{k+1}\left(\frac{1}{\varepsilon}\right)}{N^k}\right)$, where the total number of the mesh points is $\mathcal{O}\left(N^2\right)$. For $k \geq 3$, our methods can be extended directly, provided the higher order regularities of the solution u are derived. Finally, numerical experiments support our theoretical results.

1. Introduction

Consider the convection–diffusion problem

$$\begin{cases} -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $\Omega = (0, 1)^2$ and $\varepsilon > 0$ is a small parameter.

Let $\mathbf{b}, c$ and f be sufficiently smooth functions on $\bar{\Omega}$. The existence and uniqueness of the solution of (1.1) in $H^2(\Omega) \cap H_0^1(\Omega)$ are guaranteed under the additional assumptions:

$$\mathbf{b} = (b_1(x, y), b_2(x, y)) \leq -(\beta, \beta), \; c(x, y) \geq \gamma > 0, \; f(x, y) \in L_2(\Omega), \tag{1.2}$$

and

$$c_0^2 := c(x, y) - \frac{1}{2} \nabla \cdot \mathbf{b}(x, y) \geq \omega > 0. \tag{1.3}$$

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Here γ, β, ω are positive constants. Recall that the solution u of (1.1) typically has exponential boundary layers at the sides $x = 0$ and $y = 0$ of $\bar{\Omega}$ and an exponential corner layer at the point $(0, 0)$ [10].

For small values of ε, it is well known that the numerical approximation of (1.1) requires some special treatments in order to get good results. There is a lot of work dealing with numerical methods for convection–diffusion and associated problems (see for example the books [10,7] and references therein).

One of the most successful methods is the use of layer-adapted meshes appropriately refined. Layer-adapted meshes have first been proposed by Bakhvalov [1] in the context of reaction–diffusion problems, and livened up by the introduction of special piecewise-uniform meshes by Shishkin [11]. Linß [4,5] combined the ideas from Bakhvalov meshes and Shishkin meshes, and gave a novel Shishkin-type meshes. Following [5], we will call this mesh Bakhvalov–Shishkin meshes. Due to the standard Galerkin solution is very sensitive to the choice of the transition points [8,6], stabilized methods, such as the streamline diffusion finite element method (SDFEM) and the discontinuous Galerkin (DG) methods have to be considered. For 2D convection–diffusion problems, Zarin and Roos [12] obtained a uniform convergence rate of $\mathcal{O}\left((\ln N)^{1.5}/N\right)$ in associated norm by a nonsymmetric DG method with interior penalties (NIPG method) on Shishkin meshes when bilinear elements are used. Under similar assumptions, a uniform convergence rate of $\mathcal{O}(\ln N/N)$ in a DG-norm with the local discontinuous Galerkin (LDG) method was established in [15]. Furthermore, higher-order DG methods are introduced in this topic. Zhu and Zhang [16] verified a new result by the LDG method: uniform convergence of order $\mathcal{O}\left((\ln N/N)^{k+1/2}\right)$ in a DG-norm with $\mathcal{Q}_k$ elements on Shishkin meshes. Meanwhile, Zhu and Xie [14] combined the continuous Galerkin methods with the LDG method to solve 1D convection–diffusion problems, and a uniform convergence rate $\mathcal{O}\left((\ln N/N)^k\right)$ in associated norm was achieved on Shishkin meshes with piece-wise polynomial approximations of degree $k \geq 1$. Here, the total number of mesh points is $\mathcal{O}(N)$ for 1D problems and $\mathcal{O}\left(N^2\right)$ for 2D problems respectively. For more detailed discussions on this topic, we refer to the recent monograph [10] and the review article [9] by Roos for recent results in the numerical analysis.

Recently, Durán and Lombardi [2] designed a kind of graded meshes for the convection dominated convection–diffusion problems. This kind of meshes can be regarded as an alternative to the Shishkin-type meshes. Using the standard bilinear finite element method on the graded meshes, they proved optimal order error estimates in the ε-weighted H^1-norm valid uniformly, up to a logarithmic factor, in the singular perturbation parameter ε. Also, through some numerical experiments, they pointed out that the graded meshes seems to be more robust in the sense that the numerical results are not strongly affected by variations of parameters defining the meshes.

In this paper, we shall analyse the nonsymmetric DG method with interior penalties (NIPG) and the symmetric DG method with interior penalties (SIPG) on the modified graded meshes. Firstly, based on the graded meshes in Durán and Lombardi [2], we get two forms of modified graded meshes, denoted by G-meshes (I) and G-meshes (II) respectively. Then, we give the upper bound and lower bound of the mesh-sizes in G-meshes (I) and G-meshes (II) respectively. Finally, we obtain the optimal order error estimates in the ε-weighted energy norm valid uniformly, up to a logarithmic factor, in the singular perturbation parameter ε on both G-meshes (I) and G-meshes (II) with the standard Lagrange $\mathcal{Q}_k$-elements $(k = 1, 2)$. We prove that the convergence rate in the ε-weighted energy norm is $\mathcal{O}\left(\frac{\log^{k+1}\left(\frac{1}{\varepsilon}\right)}{N^k}\right)$, where the total number of the mesh points is $\mathcal{O}\left(N^2\right)$. For $k \geq 3$, our methods can be extended directly, provided the higher order regularities of the solution u are derived. Consequently, we can employ the more robust numerical methods, i.e., the NIPG method and the SIPG method on the more robust meshes, i.e., the graded meshes. Our work will complement the work in Durán and Lombardi [2], and give a feasible choice in the topic of numerical methods for the singularly perturbed problems.

The remainder of this article is structured as follows. In section 2, we introduce the modified graded meshes, i.e., G-meshes (I) and G-meshes (II). Meanwhile, we introduce the NIPG method and the SIPG method for the singularly perturbed problems. In section 3, we give the error analysis on both G-meshes (I) and G-meshes (II). This section contains our main results, while in section 4 we give some experiments to validate our theoretical results. Finally, we end in section 5 with some conclusions.

Throughout this paper, C denotes a generic positive constant that is independent of both the perturbation parameter ε and the mesh parameters h and N.

2. The graded meshes and the DG formulations

2.1. The graded meshes

Let us introduce the modified graded meshes, i.e., G-meshes (I): given a mesh-generating parameter h, and assume that it satisfies $0 < h < 1$, we introduce the partition $\{x\}_{i=0}^{N}$ of the interval $[0, 1]$ given by

$$\begin{cases} x_0 = 0, \\ x_1 = h\varepsilon, \\ x_{i+1} = x_i + hx_i, \quad \textit{for} \quad 1 \leq i \leq N-2, \\ x_N = 1, \end{cases} \tag{2.1}$$