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Fourier Collocation Algorithm for identification of a spacewise dependent source in wave equation from Neumann-type measured data

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ABSTRACT

Inverse problem of identifying the unknown spacewise dependent source $F(x)$ in $1D$ wave equation $u_{tt} = c^2 u_{xx} + F(x)G(t) + h(x,t)$, $(x,t) \in (0,1) \times (0,T)$, from the Neumann-type measured output $g(t) := u_x(0,t)$ is investigated. Most studies have attempted to reconstruct an unknown spacewise dependent source $F(x)$ from the final observation $u_T(x) := u(x,T)$. Since a boundary measured data is most feasible from an engineering viewpoint, the identification problem has wide applications, in particular, in electrical networks governed by harmonically varying source for the linear wave equation $u_{tt} - u_{xx} = F(x)cos(\omega t)$, where $\omega > 0$ is the frequency and $F(x)$ is an unknown source term. In this paper Fourier Collocation Algorithm for reconstructing the spacewise dependent source $F(x)$ is developed. This algorithm is based on Fourier expansion of the direct problem solution applied to the minimization problem for Tikhonov functional, by taking then a partial N-sum of the Fourier expansion. Tikhonov regularization is then applied to the obtained discrete ill-posed problem. To obtain high quality reconstruction in large values of the noise level, a numerical filtering algorithm is used for smoothing the noisy data. As an application, we demonstrate the ability of the algorithm on benchmark problems, in particular, on source identification problem in electrical networks governed by mono-frequency source. Numerical results show that the proposed algorithm allows to reconstruct the spacewise dependent source $F(x)$ with enough high accuracy, in the presence of high noise levels.

1. Introduction

In this paper, we study the problem of identifying an unknown spacewise dependent source $F(x)$ in

$$\begin{cases} u_{tt} = c^2 u_{xx} + F(x)G(t) + h(x,t), \quad (x,t) \in \Omega_T := (0,1) \times (0,T]; \\ u(x,0) = 0, \;\; u_t(x,0) = 0, \quad x \in (0,1); \\ u(0,t) = 0, \;\; \mu u_x(1,t) + (1-\mu)u(1,t) = 0, \quad t \in (0,T), \end{cases} \tag{1}$$

from Neumann type output (boundary measured data)

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$$g(t) := u_x(0,t), \quad t \in (0,T). \tag{2}$$

We define this problem as an inverse source problem (ISP) for wave equation (1) with Neumann type boundary measured data. Remark that, for a one-dimensional elastic string, $\tau_0 u_x(1,t)$, $\tau_0 > 0$, is an applied vertical tension at $x = 1$. The Robin boundary condition at $x = 1$, according to Hooke's law, is $\tau_0 u_x(1,t) + \kappa u(1,t) = 0$, where $\kappa > 0$ is the elastic constant of a string. Hence, the constant μ in (1) is $\mu = \tau_0/(\kappa + \tau_0) \in (0,1)$. We assume here, without loss of generality, that the initial data in (1) is zero, that is $u(x,0) = u_t(x,0) = 0$.

Inverse problems for hyperbolic equations are one of important classes of inverse problems in science and engineering [10,11,14]. These problems naturally arise from geophysical prospecting and seismology, oil and gas exploration, radar technology, electrical networks and many other physical problems (see [2,3,11–13] and references therein). An inverse source problem (ISP) of identifying an unknown source term $S(u)$ in the wave equation $u_{tt} - u_{xx} = S(u)$, $x, t > 0$, from boundary information $u(0,t) = f(t)$, $u_x(0,t) = g(t)$, has first been studied in [1]. Here an existence result for the identification problem is derived. Uniqueness results for multidimensional parabolic and hyperbolic ISPs have been established in [4] based on relations with the approximate controllability of the adjoint problems. Stability estimate and a reconstruction formula of $f(x)$ in the hyperbolic equation $u_{tt} = \Delta u + \sigma(t) f(x)$, $x \in \Omega \subset \mathbb{R}^r$, $t > 0$, from the Neumann type output data $\partial u(x,t; f)/\partial n$ has been proposed in [16]. The approach given here is based on exact boundary controllability and Volterra integral equation of the first kind with kernel $\sigma(t)$. Simultaneous identification of (interior and boundary) source terms in a linear hyperbolic problem from the final overdetermination $u_T(x) := u(x,T)$, based on weak solution approach and adjoint method, has been studied in [7].

In this paper, we develop Fourier Collocation Algorithm (FCA) for identifying the spacewise dependent source $F(x)$ in (1)–(2). For an inverse source problem related to the advection–diffusion equation $u_t = Du_{xx} - \nu u_{xx} + F(x)G(t)$ this algorithm has been proposed in [8]. This algorithm is based on Fourier expansion of solution of the direct problem (1) with subsequent use of the Nth partial sum of the Fourier expansion for the output data $u_x(0,t;F)$. Substituting then this sum in the regularized cost functional

$$J_\alpha(F) := \frac{1}{2}\|u_x(0,\cdot;F) - g\|^2_{L^2(0,T)} + \alpha\|F\|^2_{L^2(0,l)}, \quad \alpha > 0, \tag{3}$$

we then obtain a discrete system of algebraic equations which unique solution is an approximate regularized solution of the inverse problem (1)–(2). The algorithm is fast, effective and does not require solving of any integral equation. In addition, the proposed approach is that it allows also to estimate the degree of ill-posedness of the considered inverse problem. Specifically, we show here that the degrees of ill-posedness of the inverse source problems with boundary measured and final data, are almost the same. For smoothing the output data, corresponding to large values of random noise level, a numerical filtering algorithm, proposed in [8], is used. Our numerical results demonstrate that the accuracy of all reconstructions was sufficiently for high noise level random noisy data.

The paper is organized as follows. In Section 2 we introduce an input–output operator to study solvability of the regularized inverse problem. Fourier Collocation Algorithm for identification of a spacewise dependent source from Neumann type measured output data is described in Section 3. Results of computational experiments, including comparative analysis of degrees of ill-posedness of inverse source problems with boundary and final time measured data, are given in Section 4. These results illustrate high accuracy and robustness the proposed computational algorithm, as the results of numerical simulations for benchmark problems illustrate. Some concluding remarks are given in Section 5.

2. Input–output operator and solvability of the regularized problem

The measured data $g \in L^2(0,T)$ always contain a random noise, so, we will look for the unique regularized solution of the inverse problem (1)–(2). This solution $F_\alpha \in L^2(0,1)$ will be defined as a minimum of the Tikhonov functional (3).

For a given $F \in L^2(0,1)$ we denote by $u := u(x,t;F)$ a solution of the direct problem (1). It is well known that under the conditions

$$F \in L^2(0,1),\ G \in H^1(0,T),\ h_t \in L^2\left(0,T; L^2(0,1)\right) \tag{4}$$

the direct problem (1) has a unique regular weak solution $u \in C\left(0,T; H^2(0,1)\right)$, $u_t \in C\left(0,T; V(0,1)\right)$, $u_{tt} \in C\left(0,T; L^2(0,1)\right)$ [6]. Here and below $V := \{u \in H^1(0,1) : u(0,t) = 0\}$ and $H^p(0,1)$ is a Sobolev space. It follows from the main estimate (51) ([6], Ch. 7.2) that for the norm $\|u_{xx}\|_{L^2(0,1)}$ of the regular weak solution of the direct problem (1) the following estimate holds:

$$\text{ess sup}_{0\le t\le T}\|u_{xx}\|_{L^2(0,1)} \le C_0(T)\left(\|F\|_{L^2(0,1)}\|G\|_{H^1(0,T)} + \|h\|_{H^1(0,T;L^2(0,1))}\right). \tag{5}$$

We introduce the input–output operator

$$(\mathcal{K}F)(t) := (u_x(x,t;f))_{x=0}, \quad \mathcal{K} : L^2(0,1) \mapsto C(0,T), \tag{6}$$

and reformulate the inverse problem (1)–(2) as the operator equation

$$\mathcal{K}F = g,\ F \in L^2(0,1),\ g \in L^2(0,T). \tag{7}$$