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Convergent interpolatory quadrature schemes ☆

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ABSTRACT

We use a connection between interpolatory quadrature formulas and Fourier series to find a wide class of convergent schemes of interpolatory quadrature rules. In the process we use techniques coming from Riemann–Hilbert problems for varying measures and convex analysis.

1. Introduction

Let $x_{j,n}$, $j = 1, \ldots, n$, be n points such that $-1 < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < 1$. We consider the quadrature rule for continuous functions f on the interval $[-1, 1]$ ($f \in \mathscr{C}$):

$$I_n[f] = \sum_{j=1}^{n} \lambda_{j,n} f(x_{j,n}),$$

where the coefficients

$$\lambda_{j,n} = \int \frac{P_n(x)\,\mathrm{d}x}{P_n'(x_{j,n})(x - x_{j,n})}, \quad j = 1, \ldots, n, \quad \text{with} \quad P_n(x) = \prod_{j=1}^{n} \left(x - x_{j,n}\right). \tag{1}$$

In order to simplify the notation, throughout the paper we will convene that

$$\int g(x)\,\mathrm{d}x = \int_{-1}^{1} g(x)\,\mathrm{d}x.$$

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The points $x_{j,n}$, $j = 1, \ldots, n$, are called nodes of the rule, and the vector $\mathbf{x}_n = (x_{1,n}, \cdots, x_{n,n})$ is then the corresponding system of nodes. This quadrature rule satisfies the following equality for every polynomial P with degree smaller than n:

$$I_n[P] = \sum_{j=1}^{n} \lambda_{j,n} P(x_{j,n}) = \int P(x)\,\mathrm{d}x. \tag{2}$$

To see this, from Lagrange's interpolatory formula we have that

$$P(x) = \sum_{j=1}^{n} \frac{P_n(x) P(x_{j,n})}{P_n'(x_{j,n})(x - x_{j,n})},$$

and integrating, we obtain

$$\int P(x)\,\mathrm{d}x = \sum_{j=1}^{n} P(x_{j,n}) \int \frac{P_n(x)\,\mathrm{d}x}{P_n'(x_{j,n})(x - x_{j,n})} = \sum_{j=1}^{n} \lambda_{j,n} P(x_{j,n}),$$

which is (2). Those quadrature formulas are often called interpolatory quadrature rules.

Let us fix a triangular scheme of nodes $\mathbf{X} = \{\mathbf{x}_n = (x_{1,n}, \ldots, x_{n,n})\}_{n\in\mathbb{N}}$. We may wonder if the following equality holds for every continuous function $f \in \mathscr{C}$:

$$\lim_{n\to\infty} I_n[f] = \int f(x)\,\mathrm{d}x \quad \text{as} \quad n \to \infty. \tag{3}$$

In general the answer is negative. It is easy to construct a counterexample. Suppose we have a scheme with no node in a Borel set $I \subset [-1, 1]$ and $I \cap [-1, 1] \neq \emptyset$, with I of positive Lebesgue measure. We can always find a continuous function f which satisfies $f(x) = 0$ if $x \notin I$ and $f(x) > 0$ when $x \in I$. Hence for every $n \in \mathbb{N}$, $f(x_{k,n}) = 0$, $k = 1, \ldots, n$, and $\sum_{k=1}^{n} \lambda_{j,n} f(x_{k,n}) = 0$. So

$$0 = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{j,n} f(x_{k,n}) \neq \int f(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x > 0.$$

We say that the scheme of nodes $\mathbf{X}$ is convergent when (3) takes place for every $f \in \mathscr{C}$. The above example of non-convergent scheme points out that the nodes should be *well distributed* on $[-1, 1]$ in a certain sense. So now the question is: what does *well distributed* mean in this context?

A sequence of Borel measures $\{\sigma_n\}_{n\in\mathbb{N}}$ supported on $[-1, 1]$ ($\operatorname{supp}(\sigma_n) \subset [-1, 1]$) is said to be star weak convergent to another measure σ, and we denote $\sigma_n \xrightarrow{\star} \sigma$ as $n \to \infty$, if for all $f \in \mathscr{C}$ the following equality is satisfied

$$\lim_{n\to\infty} \int f(x)\,\mathrm{d}\sigma_n(x) = \int f(x)\,\mathrm{d}\sigma(x).$$

Let us introduce the sequence of signed measures $\{\mu_n\}_{n\in\mathbb{N}}$

$$\mu_n = \sum_{k=1}^{n} \lambda_{k,n} \delta_{x_{k,n}}, \qquad n \in \mathbb{N}, \tag{4}$$

where δ_x denotes Dirac's delta measure supported on x. So condition (3) can be written as $\mu_n \xrightarrow{\star} \mathrm{d}x$ as $n \to \infty$.

In [6], T. Bloom, D.S. Lubinsky, and H. Stahl found a necessary convergent condition on the distribution of nodes. Consider the sequence of zero counting measures $\{\eta_n\}_{n\in\mathbb{N}}$ corresponding to the scheme of nodes $\mathbf{X} = \{\mathbf{x}_n = (x_{1,n}, \ldots, x_{n,n})\}_{n\in\mathbb{N}}$. This means that for each $n \in \mathbb{N}$, η_n assigns mass $1/n$ to each $x_{j,n}$; explicitly:

$$\eta_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_{k,n}}, \qquad n \in \mathbb{N}. \tag{5}$$

The authors showed that if a scheme $\mathbf{X}$ converges then every weak convergent subsequence corresponding to the set of counting measures must satisfy that

$$\eta_n \xrightarrow{\star} \frac{1}{2}(\nu + \beta) = \mu, \quad \text{where} \quad \mathrm{d}\,\nu(x) = \frac{\mathrm{d}\,x}{\pi\sqrt{1 - x^2}}, \quad x \in (-1, 1), \tag{6}$$

and β is a positive and probability measure on $[-1, 1]$. Also, for any such measure μ there exists a convergent interpolatory quadrature scheme with positive weights coefficients $\lambda_{j,n}$.