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Convergence and error estimates of viscosity-splitting finite-element schemes for the primitive equations

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ABSTRACT

This paper is devoted to the numerical analysis of a first order fractional-step time-scheme (via decomposition of the viscosity) and "inf-sup" stable finite-element spatial approximations applied to the Primitive Equations of the Ocean. The aim of the paper is twofold. First, we prove that the scheme is unconditionally stable and convergent towards weak solutions of the Primitive Equations. Second, optimal error estimates for velocity and pressure are provided of order $O(k+h^l)$ for $l=1$ or $l=2$ considering either first or second order finite-element approximations (k and h being the time step and the mesh size, respectively). In both cases, these error estimates are obtained under the same constraint $k \le C h^2$.

1. Introduction

Assuming some simplifications (basically hydrostatic pressure and "rigid lid" hypothesis), the 3D Navier–Stokes problem leads to the so-called "Primitive Equations" (or Hydrostatic Navier–Stokes problem). This problem is fundamental in the field of geophysical fluids [14,28,31], because it describes the large-scale motions in the ocean [29]. The rigid lid hypothesis (vertical displacements of the free surface of the ocean are vanished) is usually assumed in Oceanography, except in the case when fast surface waves are of interest [14].

For simplicity, we take constant density and viscosity, Cartesian coordinates (x in the eastern direction, y in the northern direction and z perpendicular to the surface of the Earth) and we assume that the effects due to temperature and salinity can be decoupled from the flow dynamics. Then, the Primitive Equations model can be written as the initial–boundary problem [27,29,28]:

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$$(P)\quad\begin{cases} \partial_t\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{b}(\mathbf{u}) + \nabla_{\mathbf{x}}p = \mathbf{f} & \text{in } \Omega\times(0,T),\\ \partial_z p = -\rho\, g, \qquad \nabla\cdot\mathbf{U} = 0 & \text{in } \Omega\times(0,T),\\ \mathbf{u} = u_3 n_3 = 0 & \text{on } \Gamma_b\times(0,T),\\ \mathbf{u} = 0 & \text{on } \Gamma_l\times(0,T),\\ \nu\,\partial_z\mathbf{u} = \mathbf{g}_s, \quad u_3 = 0 & \text{on } \Gamma_s\times(0,T),\\ \mathbf{u}_{|t=0} = \mathbf{u}_0 & \text{in } \Omega, \end{cases}$$

where $\Omega = \{(\mathbf{x}, z)\in \boldsymbol{R}^3 :\ \mathbf{x} = (x, y)\in S,\ -D(\mathbf{x}) < z < 0\}$ is the 3D domain filled by the water, with $S\subset \boldsymbol{R}^2$ the surface domain (a regular bounded 2D domain) and $D : \overline{S}\to \boldsymbol{R}_+$ the bottom function (assuming $D > 0$ in S). Then, the different boundaries of Ω are denoted as $\Gamma_s = \overline{S}\times\{0\}$ the surface, $\Gamma_b = \{(\mathbf{x}, -D(\mathbf{x})) :\ \mathbf{x}\in S\}$ the bottom and $\Gamma_l = \{(\mathbf{x}, z) :\ \mathbf{x}\in\partial S, -D(\mathbf{x}) < z < 0\}$ the lateral walls (with outwards normal vector $(\mathbf{n}_{\mathbf{x}}, n_3)$).

The unknowns of the problem (P) are $\mathbf{U} = (\mathbf{u}, u_3) : \Omega\times(0, T)\to \boldsymbol{R}^3$ the 3D velocity field (with $\mathbf{u} = (u_1, u_2)$ the horizontal velocity and u_3 the vertical one) and $p : \Omega\times(0, T)\to \boldsymbol{R}$ the pressure.

In (P), $\mathbf{b}(\mathbf{u}) = f\mathbf{u}^{\perp}$ represents the effect of the Coriolis Forces, with $\mathbf{u}^{\perp} = (-u_2, u_1)^t$ and $f = 2|w|\sin\theta$, where w is the angular velocity of the Earth and $\theta = \theta(y)$ is the latitude, $\rho\in \boldsymbol{R}_+$ is the water density (that it is assumed a positive constant), $g\in \boldsymbol{R}_+$ is the gravity acceleration (another positive constant), $\mathbf{f} : \Omega\times(0, T)\to \boldsymbol{R}^2$ is a field of external horizontal forces (depending for instance on the salinity and temperature) and $\mathbf{g}_s : \Gamma_s\times(0, T)\to \boldsymbol{R}^2$ represents the stress of the wind on the surface.

Finally, $\nabla = (\nabla_{\mathbf{x}}, \partial_z)^t$ denotes the three-dimensional gradient operator (with $\nabla_{\mathbf{x}} = (\partial_x, \partial_y)^t$ its horizontal component) and $\Delta = \partial^2/\partial_{xx} + \partial^2/\partial_{yy} + \partial^2/\partial_{zz}$ the three-dimensional Laplacian operator.

Problem (P) has been vertically scaled such that the horizontal and vertical dimensions in Ω are of the same order. In fact, Ω is the non-dimensional domain obtained after a vertical scaling and problem (P) can be deduced from an asymptotic limit of the anisotropic Navier–Stokes equations when the aspect ratio (vertical vs horizontal scale) goes to zero [1,2,5].

Remark 1. When variations in the surface are important in the problem, it is usual to consider the general Navier–Stokes equations with hydrostatic pressure, introducing the free surface as a new unknown. In this case, one has to change the boundary condition of "rigid lid" ($u_3 = 0$ on Γ_s) for the equation of the free surface, arriving at the so-called three-dimensional Shallow Water model. Some numerical approximations of this model can be seen in [9,8,34].

We will give two reformulations of problem (P) leading to different spatial approximations.

By using the vertical momentum equation $\partial_z p = -\rho\, g$, the total pressure p can be decomposed as

$$p(t, \mathbf{x}, z) = p_s(t, \mathbf{x}) + p_{hyd}(z)$$

where $p_{hyd}(z) = -\rho g z$ is the hydrostatic pressure, and $p_s : S\times(0, T)\to \boldsymbol{R}$ is a new unknown (defined only on the surface S), that it will be called *surface pressure*.

Notice that incompressibility equation $\nabla\cdot\mathbf{U} = 0$ in $\Omega\times(0, T)$ and boundary condition $u_3 = 0$ on $\Gamma_s\times(0, T)$ are equivalent to the following integral formula for the vertical velocity:

$$u_3(t, \mathbf{x}, z) = \int_z^0 \nabla_{\mathbf{x}}\cdot\mathbf{u}(t, \mathbf{x}, s)\, ds. \tag{1}$$

Moreover, the following equality holds in $S\times(0, T)$:

$$\int_{-D(\mathbf{x})}^0 \nabla\cdot\mathbf{U}(t, \mathbf{x}, z)\, dz = \nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle(t, \mathbf{x}) - (\mathbf{u}, u_3)(t, \mathbf{x}, -D(\mathbf{x}))\cdot(\nabla_{\mathbf{x}}D(\mathbf{x}), 1) = 0, \tag{2}$$

where $\langle\mathbf{u}\rangle = \langle\mathbf{u}\rangle(t, \mathbf{x})$ denotes the flow rate of the water column of the horizontal velocity:

$$\langle\mathbf{u}\rangle(t, \mathbf{x}) = \int_{-D(\mathbf{x})}^0 \mathbf{u}(t, \mathbf{x}, z)\, dz.$$

Therefore, since $(\nabla_{\mathbf{x}}D(\mathbf{x}), 1)$ is parallel to the normal vector $(\mathbf{n}_{\mathbf{x}}, n_3)$ on Γ_b, assuming the incompressibility equation $\nabla\cdot\mathbf{U} = 0$ in $\Omega\times(0, T)$, the so-called slip condition $\mathbf{u}\cdot\mathbf{n}_{\mathbf{x}} + u_3 n_3 = 0$ on $\Gamma_b\times(0, T)$ is equivalent to the constraint $\nabla_{\mathbf{x}}\cdot\langle\mathbf{u}\rangle = 0$ in $S\times(0, T)$ [27–29].

Then, problem (P) can be reformulated as the following *integro-differential problem*: