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Multidomain Legendre–Galerkin Chebyshev-collocation method for one-dimensional evolution equations with discontinuity ☆

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ABSTRACT

The multidomain Legendre–Galerkin Chebyshev-collocation method is considered to solve one-dimensional linear evolution equations with two nonhomogeneous jump conditions. The scheme treats the first jump condition essentially and the second one naturally. We adopt appropriate base functions to deal with interfaces. The proposed method can be implemented in parallel. Error analysis shows that the approach has an optimal convergence rate. The proposed method is also applied to computing the one-dimensional Maxwell equation and the one-dimensional two phase Stefan problem, respectively. Numerical examples are given to confirm the theoretical analysis.

1. Introduction

The classical spectral methods are preferable to solve problems with smooth solutions on single and complex domains in [4,5]. However, they are not suitable to solve problems with jump conditions arising in scientific computations as in [14,2,3].

Various numerical methods are proposed to solve problems with jump conditions. The Yee scheme exhibits local divergence and losses of global convergence for the approximation of the discontinuous variable [8]. In [27], the implicit derivative matching method is present to restore the accuracy of high-order finite-difference time-domain methods. Some immersed finite volume and finite element methods are developed to solve elliptic interface problems in [17,11,20,12]. High-order accurate difference potentials methods for elliptic and parabolic problems with interfaces are developed in [10,1]. Multidomain pseudospectral methods are proposed for solving nonlinear convection–diffusion equation in [15].

In this paper, we consider the following parabolic equation with two nonhomogeneous jump conditions as

$$\begin{cases} \partial_t U - \partial_x(\epsilon \partial_x U) = f(x,t), & x \in I_1 \cup I_2, t \in (0,T], \\ [U]_0 = \alpha, [\epsilon \partial_x U]_0 = \beta, & t \in (0,T], \\ U(-1,t) = U(1,t) = 0, & t \in [0,T], \\ U(x,0) = U_0(x), & x \in I, \end{cases} \tag{1.1}$$

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where $I_1 = (-1, 0), I_2 = (0, 1)$ and $I = (-1, 1)$, $\epsilon|_{I_i} = \epsilon_i$ is positive piecewise constant, the jump is defined by $[v]_0 = v(0+) - v(0-)$, and α, β are constants. Such jump conditions arise in many areas [14,2,12]. The multidomain Legendre–Galerkin Chebyshev-collocation (MLGCC) method is developed to solve the problem (1.1). The scheme is based on the Legendre method, but the right and initial terms are collocated by the Chebyshev–Gauss–Lobatto (CGL) points. The Crank–Nicolson method is employed for the time discretization. The scheme treats the first jump condition essentially and the second one naturally. As in [15], the appropriate base functions are constructed to deal with the interface for solving the problem (1.1) in parallel. The stability and the optimal rate of convergence are derived. Applications of the MLGCC method to the one-dimensional (1D) Maxwell equation and the 1D two phase Stefan problem are considered.

The article is organized as follows. In Section 2, some notations and the scheme can be presented. In Section 3, results on approximation are given. In Section 4, we prove the stability and convergence of the fully discrete scheme. Some corresponding numerical results are given. In Section 5 and 6, we use our method to solve the 1D Maxwell equation and the 1D two phase Stefan problem, and numerical results also are presented.

2. Notations and schemes

In the section, some notations and the MLGCC scheme are presented. Let $(\cdot, \cdot)_J$ and $\|\cdot\|_J$ be the inner product and the norm of the space $L^2(J)$, respectively. For any non-negative integer $\sigma > 0$, we adopt the standard notation $H^\sigma(J)$ for the Sobolev space equipped with the norm $\|\cdot\|_{\sigma,J}$ and the semi-norm $|\cdot|_{\sigma,J}$. We drop the subscript J whenever $J = I$. Let $H^{-1}(I) = (H_0^1(I))'$ be the dual space. Denote by $\hat{x}_j^i$ the CGL nodes on $\hat{I} = (-1, 1)$ and we set $a_0 = -1, a_1 = 0, a_2 = 1$. Define $h_i = a_i - a_{i-1}$ and

$$I_N^i = \{x_j^i : x_j^i = \frac{h_i \hat{x}_j^i + a_{i-1} + a_i}{2},\ 0 \le j \le N_i,\ i = 1, 2\}. \tag{2.1}$$

In this paper, we shall use the piecewise Sobolev spaces. Let $u_i := u|_{I_i}$ and define

$$\tilde{H}^\sigma(I) = \{u : u|_{I_i} \in H^\sigma(I_i), i = 1, 2\},$$
$$\tilde{H}^1_{0,\square} = \{u \in \tilde{H}^1(I) : u(-1) = u(1) = 0, [u]_0 = \alpha\},$$

with the broken semi-norm

$$|u|_{\tilde{H}^\sigma(I)} = (\sum_{i=1,2} |u|^2_{\sigma,I_i})^{1/2}.$$

Let $\mathbb{P}_{N_i}$ be the space of polynomials of the degree at most N_i. The piecewise polynomial spaces are defined as

$$\begin{aligned} V_N^\square &= \{\varphi \in \tilde{H}^1_{0,\square}(I) : \varphi|_{I_i} \in \mathbb{P}_{N_i}, i = 1, 2\}, \\ V_N &= \{\varphi \in H_0^1(I) : \varphi|_{I_i} \in \mathbb{P}_{N_i}, i = 1, 2\}. \end{aligned} \tag{2.2}$$

The problem (1.1) can be written in a weak form: find $U(t) \in \tilde{H}^1_{0,\square}(I)$ such that for any $V \in H_0^1(I)$,

$$\begin{cases} (\partial_t U, V) + \sum\limits_{i=1,2} (\epsilon \partial_x U, \partial_x V)_{I_i} = (f, V) - \beta V(0), & t > 0, \\ U(x, 0) = U_0(x), & x \in I, \end{cases} \tag{2.3}$$

where the first jump condition is treated essentially and the second one naturally. The semi-discrete Legendre–Galerkin approximation is to find $u_N \in V_N^\square$ such that for any $\varphi \in V_N$,

$$\begin{cases} (\partial_t u_N, \varphi) + \sum\limits_{i=1,2} (\epsilon \partial_x u_N, \partial_x \varphi)_{I_i} = (I_N^C f, \varphi) - \beta \varphi(0), & t > 0, \\ u_N(x, 0) = I_N^C U_0(x), & x \in I, \end{cases} \tag{2.4}$$

where I_N^C is the Chebyshev interpolation operator such that

$$(I_N^C v)|_{I_i}(x_j^i) = v|_{I_i}(x_j^i), \quad x_j^i \in I_N^i, \quad 0 \le j \le N_i, \quad i = 1, 2. \tag{2.5}$$

Let τ be the mesh size in variable t and set $t_k = k\tau$, $k = 0, 1, \cdots, n_T$ $(n_T \tau = T)$. For simplicity, we denote $u^k(x) := u(x, t_k)$ by u^k and define

$$u_t^k = \frac{u^{k+1} - u^k}{\tau}, \quad \bar{u}^k = \frac{u^{k+1} + u^k}{2}. \tag{2.6}$$

The Crank–Nicolson method is applied to (2.4) in the time discretization. Thus, the fully discrete scheme is to find $u_N^k \in V_N^\square$ such that for any $\varphi \in V_N$,