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Accurate cubature and extended Kalman filtering methods for estimating continuous-time nonlinear stochastic systems with discrete measurements

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A B S T R A C T

This paper further advances the idea of accurate Gaussian filtering towards efficient cubature Kalman filters for estimating continuous-time nonlinear stochastic systems with discrete measurements. It implies that the moment differential equations describing evolution of the predicted mean and covariance of the propagated Gaussian density in time are solved accurately, i.e. with negligible error. The latter allows the total error of the cubature Kalman filtering to be reduced significantly and results in a new accurate continuous–discrete cubature Kalman filtering method. At the same time, we revise the earlier developed version of the accurate continuous–discrete extended Kalman filter by amending the involved iteration and relaxing the utilized global error control mechanism. In addition, we build a mixed-type method, which unifies the best features of the accurate continuous–discrete extended and cubature Kalman filters. More precisely, the time updates are done in this state estimator as those in the first filter whereas the measurement updates are conducted with use of the third-degree spherical-radial cubature rule applied for approximating the arisen Gaussian-weighted integrals. All these are examined in severe conditions of tackling a seven-dimensional radar tracking problem, where an aircraft executes a coordinated turn, and compared to the state-of-the-art cubature Kalman filters.

1. Introduction

Mathematical models in science and engineering often include an uncertainty component. They are written in the form of Itô-type *stochastic differential equation* (SDE)

$$dx(t) = F\big(x(t), u(t)\big)dt + G(t)dw(t), \quad t > 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the n-dimensional vector of system's state at time t, $u(t) \in \mathbb{R}^p$ is the vector of time-variant optional known parameters at time t (these may also include control inputs at time t), $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a nonlinear function representing the dynamic behavior of the model, $G(t)$ is a known variable matrix of size $n \times q$ and $\{w(t), t > 0\}$ is a Brownian motion with square diffusion matrix $Q(t) > 0$ of size q [3,5,11,12,18,41,46]. As customary, here and below, the notation $Q(t) > 0$ indicates that the matrix $Q(t)$ is positive definite. The initial state x_0 of SDE (1) is supposed to be a

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random variable, i.e. $x_0 \sim \mathcal{N}(\bar{x}_0, \Pi_0)$ with $\Pi_0 > 0$, where the notation $\mathcal{N}(\bar{x}_0, \Pi_0)$ stands for the normal distribution with mean $\bar{x}_0$ and covariance Π_0.

The task of state estimation in mathematical model (1) is to combine real measurements of some system's variables or their function (depending on the utilized technology) with computation of remaining (not measurable) parameters by means of an appropriate nonlinear filtering algorithm. It is usually assumed that the measurement information arrives discretely and in equidistant intervals of size $\delta = t_k - t_{k-1}$. This time interval δ is called the *sampling period* (or *waiting time*) in filtering theory. Actually, the sampling may be irregular in practice, and all our variable-stepsize Kalman-like filters cover that case of state estimation problems with no change. Nevertheless, the equidistant fashion of the sampling accepted in this paper is important for the state-of-the-art fixed-stepsize cubature Kalman filters also studied below.

The state estimation task under consideration establishes that each measurement z_k is linked to the corresponding state vector x_k of SDE (1) by the formula

$$z_k = h(x_k) + v_k, \quad k \ge 1, \tag{2}$$

where k stands for a discrete time index (i.e. x_k refers to the true state $x(t_k)$ at time t_k), $z_k \in \mathbb{R}^m$ is the information available at time t_k, $h : \mathbb{R}^n \to \mathbb{R}^m$ is a sufficiently smooth function and the measurement noise $v_k \sim \mathcal{N}(0, R_k)$ with $R_k > 0$. All realizations of the noises $w(t)$, v_k and the initial state x_0 are assumed to be taken from mutually independent Gaussian distributions.

Among others, the *cubature Kalman filtering* (CKF) is a newly devised and interesting method for estimating stochastic systems. It originates from the variant designed for treating discrete-time models in [1] and, then, it is extended to the continuous–discrete stochastic state-space system (1), (2) in [2]. The main idea behind this method is to apply the third-degree spherical-radial cubature rule for approximating Gaussian-weighted integrals. It is also shown that the CKF is a particular case of the unscented Kalman filtering (UKF) with specially selected parameters [1,2].

Särkkä and Solin [37] study various CKF implementations, which can be constructed in the frame of *linearized discretization* and *discretized linearization* approaches. These approaches are distinguished in the order of operations of linearization and discretization used for designing CKF versions. The same classification of state estimators but built in the frame of extended Kalman filtering (EKF) is considered in [8,13,22]. Notice that Frogerais et al. [8] name the discussed approaches as the *discrete–discrete* and *continuous–discrete* EKFs, respectively. Most importantly, in contrast to conclusions made in [13, 37], Kulikov and Kulikova [22,24,26] provide some evidence that filters constructed in the second approach are more accurate and preferable for practical use. This is because the discretization error committed in the first approach is made in a stochastic setting, i.e. it is a random variable. Therefore it is even unclear in what sense such an error should be evaluated and regulated (i.e. reduced if necessary). Eventually, the mentioned discretization error is not under the user's automatic control and state estimators of such sort are implemented on equidistant meshes. All this means that they return the predicted mean and covariance of the propagated Gaussian distribution with unpredictable errors, which can fail the filtering procedure. In contrast, the discretization error of the second approach is committed in a deterministic setting and, hence, it can be easily regulated by existing ODE solvers with automatic error control. The latter increases the accuracy and robustness of state estimation and allows the continuous–discrete models (1), (2) with even sparse measurements to be treated as well [22,24,26]. In addition, it is known that different stochastic discretization schemes utilized in the discrete–discrete Kalman-like filtering may lead to different interpretations of the same algorithm [7]. We stress that such different interpretations are impossible in filters built in the frame of the second approach because their discretizations are fulfilled in the deterministic setting, where all numerical solutions converge to the unique exact one under certain smoothness conditions. That is why we support the superiority of the continuous–discrete CKF and construct our new state estimators within this framework.

Below, following the continuous–discrete approach, we extend the above-cited idea of accurate Gaussian filtering towards cubature Kalman-like state estimators and devise the new *accurate continuous–discrete cubature Kalman filter* (ACD-CKF) grounded in accurate numerical integrations of the *moment differential equations* (MDEs) obtained in [37]. Moreover, taking into account the higher efficiency of the embedded pair `NIRK6(4)` with global error control in comparison to the adaptive method `NIRK4(2)` exhibited in [20], we restrict ourselves to the first integrator for implementing our ACD-CKF. We improve also the *accurate continuous–discrete extended Kalman filter* (ACD-EKF) methods constructed in [22,24,26] by amending the involved iteration and relaxing the utilized global error control mechanism. In addition, we present a mixed-type state estimator, which unifies the best features of the ACD-EKF and CKF. More precisely, the time updates in this method are done as those in the first filter whereas the measurement updates are calculated with use of the third-degree spherical-radial cubature rule applied for approximating the arisen Gaussian-weighted integrals. All these are examined in severe conditions of tackling a seven-dimensional radar tracking problem, where an aircraft executes a coordinated turn, and compared to the state-of-the-art fixed-stepsize *continuous–discrete cubature Kalman filter* (CD-CKF) in [2].

2. Square-root accurate continuous–discrete cubature Kalman filter

Given a continuous–discrete stochastic system (1), (2) and with use of equivalence of the UKF and CKF proved in [1,2], Särkkä and Solin [37] extend the earlier developed CD-UKF of Särkkä [36] towards the CD-CKF method by means of the following deterministic MDEs: