Applied Numerical Mathematics 110 (2016) 26–40

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A fixed grid, shifted stencil scheme for inviscid fluid–particle interaction

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ARTICLE INFO

Article history:
Received 3 February 2015
Received in revised form 8 June 2016
Accepted 9 August 2016
Available online 12 August 2016

Keywords:
Solid–fluid interaction
Burgers equation
Finite volume scheme
Singular source term
Moving mesh scheme
Well-balanced scheme
Moving interface
PDE-ODE coupling

ABSTRACT

This paper presents a finite volume scheme for a scalar one-dimensional fluid–particle interaction model. When devising a finite volume scheme for this model, one difficulty that arises is how to deal with the moving source term in the PDE while maintaining a fixed grid. The fixed grid requirement comes from the ultimate goal of accommodating two or more particles. The finite volume scheme that we propose addresses the moving source term in a novel way. We use a modified computational stencil, with the lower part of the stencil shifted during those time steps when the particle crosses a mesh point. We then employ an altered convective flux to compensate the stencil shifts. The resulting scheme uses a fixed grid, preserves total momentum, and enforces several stability properties in the single-particle case. The single-particle scheme is easily extended to multiple particles by a splitting method.

1. Introduction

This paper concerns a one-dimensional model of fluid–structure interaction proposed in [10]:

$$\begin{cases} u_t + \partial_x(u^2/2) = \lambda \left(h'(t) - u\right) \delta(x - h(t)), & (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ mh''(t) = \lambda \left(u(h(t),t) - h'(t)\right), & t \in \mathbb{R}_+ \\ u(x,0) = u_0(x), \quad (h(0), h'(0)) = (h_0, v_0). \end{cases} \tag{1.1}$$

Here $\delta(x)$ denotes the Dirac delta measure concentrated at $x = 0$. The function $u = u(x,t)$ models the velocity of the fluid, $h(t)$ models the location of a particle at time t, $\lambda > 0$ is a drag coefficient, and $m > 0$ is the mass of the particle.

The fluid velocity is governed by the inviscid Burgers equation, and the particle–fluid coupling is due to friction, more specifically the drag term $\lambda \left(u - h'\right)$ which appears in both the PDE and the ODE in (1.1). Since there is no viscosity, the velocity $u(x,t)$ admits entropy weak solutions, meaning that shock waves occur. This leads to complex interactions between the resulting shock wave and the particle. The model is readily extended (at least formally) to accommodate multiple particles, and then there are interesting features of the solutions that include particles drafting and passing by one another.

The model (1.1) presents several conceptual and computational difficulties. First is the singular source term on the right side of the PDE in (1.1). Because there is generally a jump in the velocity u at the location of the particle $x = h(t)$, the source term is not a distribution. Next, the ODE governing the particle motion has a discontinuous right hand side. Finally, and this

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http://dx.doi.org/10.1016/j.apnum.2016.08.002
0168-9274/

is the focus of the present paper, is the fact that the source term is moving. From a computational point of view, a potential method of dealing with this is to use a moving grid, so that the particle is always located at a grid cell boundary. However, it is not likely that this approach extends readily to the case where there is more than one particle, especially when the particle paths intersect. For this reason, a method that uses a fixed grid is desirable.

The model (1.1) has been studied in detail in a series of papers [3,5–7,10]. In [10] Lagoutière, Seguin and Takahashi provide a definition of solutions for (1.1) by studying two regularizations. They use a viscous regularization, which results in entropy inequalities, and they mollify the delta function, which leads to the proper interpretation of the nonconservative product. With these definitions in hand they completely solve the Riemann problem for (1.1), and describe the asymptotic behavior of solutions.

In [5], Andreianov, Lagoutière, Seguin and Takahashi propose a definition of entropy solution for (1.1), address the well-posedness of the problem, and introduce two finite volume methods for computing approximate solutions. One is a Glimm-like scheme, and the other is a well-balanced scheme that uses nonrectangular space–time cells near the interface. Both of the finite volume methods employ random sampling for placing the particle at a mesh interface at each time step. The nonconservative source term is handled by using a certain well-balanced scheme that was analyzed in [7]. The proper coupling of the ODE to the PDE results by enforcing a conservation of momentum principle. With these techniques they avoid the use of a moving mesh, and also avoid the use of a Riemann solver for the full model. A splitting technique is employed in order to accommodate multiple particles.

In [7], Andreianov and Seguin study in detail the model

$$u_t + \left(u^2/2\right)_x = -\lambda u \delta(x), \quad u(x,0) = u_0(x). \tag{1.2}$$

This can be viewed as a simplification of the full model (1.1), where the particle is stationary. Its analysis is an important step in understanding (1.1), due to the presence of the nonconservative product on the right side. In order to prove existence, and for the purpose of practical computation of solutions, the authors construct a finite volume scheme, which is the one that we use as the starting point for our new scheme for (1.1). In order to establish well-posedness, the authors use the theory of conservation laws with discontinuous flux [4].

In [6], Andreianov, Lagoutière, Seguin and Takahashi prove well-posedness of the model (1.1), assuming that the initial data is of bounded variation. A wave-front tracking algorithm is used to generate approximate solutions, and among other things, a BV estimate is established for the approximations.

In [3], Aguillon, Lagoutière and Seguin propose a class of finite volume schemes for (1.1). The schemes are similar to those in [5], the important difference being that a moving grid is used, in order to keep the particle located at a fixed cell boundary. The authors are able to provide a proof of convergence to the unique entropy solution of (1.1).

Very recently, a generalized version of (1.1), where the fluid is governed by the inviscid compressible Euler equations, has been studied by Aguillon [2,1].

In this paper we follow [5], starting from the same well-balanced scheme for (1.2), coupling the ODE to the PDE via conservation of momentum, and using a splitting method to accommodate two or more particles. Our contribution is an alternative method of handling the moving source term. We use a modified computational stencil, with the lower part of the stencil shifted during those time steps when the particle crosses a mesh point. We then employ an altered convective flux to compensate the stencil shifts. The resulting scheme uses a fixed grid, preserves the total momentum of the system, and for the single-particle model, it enforces a bound on the total variation of the solution. By testing the new scheme against Riemann problems (where the solutions are known from [10]), we find that our new scheme produces approximations that seem to converge to the correct solutions as the mesh size shrinks.

The remainder of this paper is organized as follows. In Section 2, we present our scheme for the case of a single particle, and then prove several stability properties of the scheme. In Section 3, we describe our splitting algorithm, which extends the single-particle scheme to the case of two particles. In Section 4, we describe a number of numerical experiments, the results of which indicate that our new method produces approximate solutions that are consistent with the physically relevant ones discussed in [3,5,6,10]. Section 5 is a brief conclusion.

2. Single particle

We use a uniform spatial mesh size Δx, and temporal step size Δt^n that can be variable. Define

$$x_j = j\Delta x,, \quad j \in \mathbb{Z}, \quad t^0 = 0, \quad t^{n+1} = t^n + \Delta t^n, \quad n \ge 0, \tag{2.1}$$

and let $\mu^n = \Delta t^n/\Delta x$. We denote by U_j^n the finite-difference approximation of $u(x_j, t^n)$, and

$$U^n := \left(\dots, U_{-2}^n, U_{-1}^n, U_0^n, U_1^n, U_2^n, \dots\right), \quad \left\|U^n\right\|_\infty := \sup_{j\in\mathbb{Z}} \left|U_j^n\right|. \tag{2.2}$$

We will use the following finite difference notation:

$$\Delta_+ U_j^n = U_{j+1}^n - U_j^n, \quad \Delta_- U_j^n = U_j^n - U_{j-1}^n. \tag{2.3}$$