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A reconstructed central discontinuous Galerkin-finite element method for the fully nonlinear weakly dispersive Green–Naghdi model

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ABSTRACT

In this paper, we present a class of high order reconstructed central discontinuous Galerkin-finite element methods for the fully nonlinear weakly dispersive Green–Naghdi model, which describes a large spectrum of shallow water waves. In the proposed methods, we first reformulate the Green–Naghdi model into conservation laws coupled with an elliptic equation, and then discretize the conservation laws with reconstructed central discontinuous Galerkin methods and the elliptic equation with continuous FE methods. The reconstructed central discontinuous Galerkin methods can be viewed as a class of fast central discontinuous Galerkin methods, in which we replace the standard formula for the numerical solution defined on the dual mesh in the central discontinuous Galerkin method with a projection equation in the L^2 sense. The proposed methods reduce the computational cost of the traditional methods by nearly half but still maintain the formal high order accuracy. We study the L^2 stability and an L^2 *a priori* error estimate for smooth solutions of the reconstructed central discontinuous Galerkin method for linear hyperbolic equation. Numerical tests are presented to illustrate the accuracy and computational efficiency of the proposed method.

1. Introduction

The so-called Green–Naghdi models [18,8] are a class of fully nonlinear weakly dispersive shallow water equations, which are applicable to a wide range of problems involving small- to large-amplitude waves on both shallow and relatively deep water. Therefore, many researchers are interested in the development and simulation of Green–Naghdi models in recent years [1,2,4–6,9,17]. We also developed a central discontinuous Galerkin-finite element (CDG-FE) method for the Green–Naghdi model over flat bottom topography and a well-balanced CDG-FE method for the Green–Naghdi model over variable bottom topographies [14].

The underlying central discontinuous Galerkin (CDG) methods, originally proposed by Liu and his collaborators for hyperbolic conservation laws [15], are a family of high order numerical methods defined on overlapping meshes. These methods can be systematically formulated with any order of (formal) accuracy and do not employ any numerical flux at element interfaces as in discontinuous Galerkin (DG) methods by evolving two sets of numerical solutions. Therefore, the CDG meth-

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ods have been widely applied to various equations, such as diffusion equations [16], Hamilton–Jacobi equations [12], ideal magnetohydrodynamics equations [11,13], Green–Naghdi equations [14] and Camassa–Holm equation [10].

However, the CDG methods and the CDG-FE methods are time-consuming and use more storage space due to evolving two sets of numerical solutions, compared with the DG methods. For reducing the computational cost of the CDG-FE methods, we present a reconstructed CDG-FE (RCDG-FE) method for the Green–Naghdi model over flat bottom topography. In the RCDG-FE method, we first reformulate the Green–Naghdi model into conservation laws coupled with an elliptic equation as in [14], and then discretize the conservation laws with the reconstructed CDG (RCDG) methods and the elliptic equation with the FE methods. In RCDG methods, we reconstruct the approximate solution defined on the dual mesh by a projection equation in the L^2 sense, so we do not need to evolve the numerical solution defined on the dual mesh by the standard formula in the CDG methods. Since the computational cost of the projection equation is far less than that of the standard formula in the CDG methods and we do not need to calculate the numerical solution on the primal mesh for the elliptic equation in the RCDG-FE methods, the RCDG-FE methods reduce the computational cost of the CDG-FE methods by nearly half but still maintain the high order accuracy.

The remainder of the paper is organized as follows. In section 2, we review the mathematical formulation of the Green–Naghdi model over flat bottom topography and its reformulation. Section 3 is devoted to the numerical methods, including the CDG-FE method and the RCDG-FE method. Then, we study the L^2 stability and an L^2 *a priori* error estimate for smooth solutions of the reconstructed central discontinuous Galerkin method for linear hyperbolic equation in section 4. In section 5, a set of numerical experiments are presented to illustrate the accuracy and computational efficiency of the RCDG method. Finally, concluding remarks are given in section 6.

2. Governing equations

We consider the fully nonlinear weakly dispersive shallow water waves by the Green–Naghdi model over flat bottom topography in one-dimensional space [18],

$$\begin{cases} h_t + (hu)_x = 0\,, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2 + \frac{1}{3}h^3(u_x^2 - u_{xt} - uu_{xx})\right)_x = 0\,, \end{cases} \tag{1}$$

where h is the total water depth, u is the vertically averaged horizontal velocity, g is the gravitational constant. The subscripts t and x denote the partial derivatives with respect to the time variable t and the spatial variable x, respectively.

The model (1) can be reformulated as conservation laws [3,9,14]

$$\begin{cases} h_t + (hu)_x = 0\,, \\ (hK)_t + \left(hKu + \frac{1}{2}gh^2 - \frac{2}{3}h^3u_x^2\right)_x = 0 \end{cases} \tag{2}$$

coupled with an elliptic equation

$$-\frac{1}{3}\left(h^3u_x\right)_x + hu = hK\,. \tag{3}$$

We shall design numerical schemes for this form of the Green–Naghdi model in the following section. For ease of presentation, we rewrite (2) as

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}, u)_x = 0\,, \tag{4}$$

with $\mathbf{U} = (h, hK)^\top$ and

$$\mathbf{F}(\mathbf{U}, u) = \left(hu, hKu + \frac{1}{2}gh^2 - \frac{2}{3}h^3u_x^2\right)^\top$$

being the flux.

3. Numerical schemes

In this section, we develop numerical schemes for the reformulated Green–Naghdi model (3) and (4). Let $\{x_{j-\frac{1}{2}}\}_j$ be a uniform partition of the computational domain $\Omega = [x_{min}, x_{max}]$ with mesh size Δx. With $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$, $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and $I_{j+\frac{1}{2}} = (x_j, x_{j+1})$, we define two discrete function spaces, associated with overlapping meshes $\{I_j\}_j$ (the primal mesh) and $\{I_{j+\frac{1}{2}}\}_j$ (the dual mesh), to approximate $\mathbf{U}$,

$$\mathcal{V}_h^C = \mathcal{V}_h^{C,k} = \{\mathbf{v} : \mathbf{v}|_{I_j} \in [P^k(I_j)]^2\,, \forall\, j\}\,,$$
$$\mathcal{V}_h^D = \mathcal{V}_h^{D,k} = \{\mathbf{v} : \mathbf{v}|_{I_{j+\frac{1}{2}}} \in [P^k(I_{j+\frac{1}{2}})]^2\,, \forall\, j\}\,,$$