Applied Numerical Mathematics 110 (2016) 128–147

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

# Two-point boundary value problems associated to functional differential equations of even order solved by iterated splines

Alexandru Mihai Bica *, Mircea Curila, Sorin Curila

*Department of Mathematics and Informatics, Faculty of Environment Protection, Faculty of Electrical Engineering and Information Technology, University of Oradea, Universitatii Street no. 1, 410087 Oradea, Romania*

ARTICLE INFO

*Article history:*
Received 15 December 2015
Accepted 14 August 2016
Available online 26 August 2016

*Keywords:*
Two-point boundary value problems
Functional differential equations of even order
Iterative numerical method
Natural cubic splines

ABSTRACT

A new iterative numerical method to solve two-point boundary value problems associated to functional differential equations of even order is proposed. The method uses a cubic spline interpolation procedure activated at each iterative step. The convergence of the method is proved and it is tested on some numerical experiments. The notion of numerical stability with respect to the choice of the first iteration is introduced proving that the proposed method is numerically stable in this sense.



## 1. Introduction

The study of two-point boundary value problems for even order differential equations is motivated by its applications in various fields of Engineering, Physics and Astrophysics (for instance, problems regarding the theory of mechanic vibrations, hydromagnetic stability (see [18]), and thermal instability). In this paper we propose a new iterative numerical method for solving such problems with deviating argument of the form:

$$\begin{cases} x^{(2p)}(t) = f(t, x(t), x(\varphi(t))), & t \in [a,b] \\ x^{(i)}(a) = a_i, \quad x^{(i)}(b) = b_i, & i = \overline{0, p-1} \end{cases} \tag{1}$$

where $\varphi : [a,b] \to \mathbb{R}$, $a \le \varphi(t) \le b$, $\forall t \in [a,b]$. The existence of solutions for two-point boundary value problems is investigated by using the corresponding Green function (see [3,4,19,20]), or the Ekeland variational principle (see [45]), or by applying the Krasnoselskii's fixed point theorem (see [72]). Interesting and well-known particular cases are obtained for $p = 1$ and $p = 2$, that are extensively studied. The case $p = 2$ corresponds to the clamped beam equation. For the clamped beam nonlinear equation, the existence of positive solutions is investigated in [1] using the contraction principle, in [44] using the topological degree and fixed point theory of nonlinear operators in lattices, and in [72] using the Krasnoselski-i's fixed point theorem of cone expansion–compression type. In [71] there are established lower and upper estimates of the positive solutions. In [36] the existence of positive solution of the equation $x^{IV}(t) = h(t) \cdot f(t, x(t), x(\varphi(t)))$, $t \in [0,1]$, is obtained using the fixed point theorem of Avery and Peterson (see [12]). In [43] even order differential equations are approached by using a fixed point theorem presented by Granas et al. in [30].

* Fax: +40 0259 408 461.
*E-mail address:* abica@uoradea.ro (A.M. Bica).

http://dx.doi.org/10.1016/j.apnum.2016.08.003

There are several numerical methods for second order two-point boundary value problems with deviating argument that use various techniques, such as: finite differences, Taylor series, shooting techniques, spectral methods (like tau, Galerkin, and collocation), spline functions, iterative methods (see [2,15,16,21,40,46,51,66]). For the numerical solution of fourth order equations and specially for the clamped beam equation, we can mention the following methods: iterative techniques in [1], [64], and [65], analytic-numeric methods like Adomian decomposition and differential transform (see [10] and [28]), shooting method in [11], finite element method in [13], finite differences method in [68], Padé approximants (see [67]), Haar wavelets method (see [54]), spectral-Galerkin method in [26], collocation method in [49], and the polynomial and non-polynomial spline functions methods in [35,50,53,55,69]. Fourth order functional differential equations of the form $x^{IV}(t) = f(t, x(t), x(\varphi(t)))$ are approached in [36] (for continuous function $\varphi$), and in [64], and [65] for the case of constant delay and anticipation. In [64] and [65] the numerical solution is obtained by using a quasilinear iterative method and the Picard's iterative method, respectively.

The numerical methods developed for high even order boundary value problems, like sixth, eighth, tenth and twelfth order, involve various techniques such as: reproducing kernel space method (see [9,8]), spline functions techniques (see [6,41,56,58]), analytic-numeric methods like Adomian decomposition, homotopy analysis, variational iteration, optimal homotopy asymptotic, and homotopy perturbation (see [31,33,37,42,47,48,70]), polynomial and non-polynomial splines (see [7,57,60,59,61,62]), Galerkin techniques (see [38]), finite element techniques (see [39]), and the differential transform method (see [29] and [34]).

The existing numerical methods constructed for $2p$th-order two-point boundary value problems are based on: Haar wavelets (see [54]), Bernstein–Petrov–Galerkin techniques (see [22]), ultraspherical-Galerkin algorithm (see [23]), Chebyshev–Galerkin projection (see [24]), Bernstein–Galerkin approximation (see [25]), Sinc-Galerkin method (see [27]), and non-polynomial splines (see [52]).

In this paper we develop the method of iterated splines for the two-point boundary value problem (1) based on the interpolation procedure of natural cubic splines that is activated at each iterative step interpolating the values computed in the previous step.

The paper is organized as follows: in Section 2 we present the iterative method to compute the approximate solution of (1) and the convergence of this method is proven in Section 3 by providing the error estimate and investigating the numerical stability with respect to the boundary values. Two interesting particular cases corresponding to second and fourth order differential equations are studied in Section 4, obtaining specific error estimates. Some numerical experiments are presented in Section 5 in order to illustrate the accuracy of the method and to test the theoretical results. Finally, we point out some concluding remarks regarding to the effectiveness and to the limitations of the proposed method.

## 2. Constructing the iterative algorithm

For given mesh

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

and for given values $y_0, y_1, \ldots, y_n \in \mathbb{R}$, the natural cubic spline $s \in C^2[a, b]$ interpolating these values has the restrictions $s_i$ to the intervals $[x_{i-1}, x_i]$, $i = \overline{1, n}$:

$$s_i(x) = \frac{(x_i - x)^3}{6h_i} \cdot M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} \cdot M_i + \\ + \left( y_{i-1} - \frac{M_{i-1} h_i^2}{6} \right) \cdot \frac{x_i - x}{h_i} + \left( y_i - \frac{M_i h_i^2}{6} \right) \cdot \frac{x - x_{i-1}}{h_i}, \quad x \in [x_{i-1}, x_i], \tag{2}$$

where $h_i = x_i - x_{i-1}$, $i = \overline{1, n}$, and $M_i = s''(x_i)$, $i = \overline{0, n}$, with $s''(x_0) = s''(x_n) = 0$, according to the natural boundary conditions. When $y_i$, $i = \overline{0, n}$, are the values of a continuous function $f : [a, b] \to \mathbb{R}$ on the mesh, that is $y_i = f(x_i)$, $\forall i = \overline{0, n}$, we say that $s$ interpolates $f$. In the construction of the iterative algorithm we will apply this natural cubic spline interpolation procedure at each iterative step, interpolating the values computed in the previous step on the knots of a uniform mesh.

Now, consider a uniform partition of $[a, b]$ with the knots $t_i = a + \frac{i \cdot (b-a)}{n}$, $i = \overline{0, n}$. Let $g : [a, b] \to \mathbb{R}$ be the two-point Hermite type interpolation polynomial of degree $2p - 1$ generated by the interpolation conditions:

$$g^{(i)}(a) = a_i, \quad g^{(i)}(b) = b_i, \quad i = \overline{0, p-1}$$

and $G : [a, b] \times [a, b] \to \mathbb{R}$ be the Green function corresponding to the boundary value problem (1),

$$G(t, s) = \begin{cases} H(t, s), & a \le s \le t \le b \\ K(t, s), & a \le t \le s \le b. \end{cases}$$

The steps of the iterative algorithm for approximating the solution of the boundary value problem (1) on the knots $t_i$, $i = \overline{0, n}$, are the following:

*Step 1*: $x_0(t) = g(t)$, $\forall t \in [a, b]$;