Applied Numerical Mathematics 110 (2016) 174–189

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

The inexact-Newton via GMRES subspace method without line search technique for solving symmetric nonlinear equations ☆

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ARTICLE INFO

Article history:
Received 6 June 2015
Received in revised form 11 August 2016
Accepted 23 August 2016
Available online 30 August 2016

Keywords:
Inexact-Newton method
GMRES method
Nonlinear equations
Global convergence

ABSTRACT

In this paper, we propose an inexact-Newton via GMRES (generalized minimal residual) subspace method without line search technique for solving symmetric nonlinear equations. The iterative direction is obtained by solving the Newton equation of the system of nonlinear equations with the GMRES algorithm. The global convergence and local superlinear convergence rate of the proposed algorithm are established under some reasonable conditions. Finally, the numerical results are reported to show the effectiveness of the proposed algorithm.

1. Introduction

In this paper, we focus on the numerical solution of systems of symmetric nonlinear equations:

$$g(x) = 0, \quad x \in \mathbf{R}^n, \tag{1.1}$$

where $g : \mathbf{R}^n \to \mathbf{R}^n$ is continuously differentiable. We will focus our attention on the case where the Jacobian $H(x) = \nabla g(x)$ of g is symmetric for all $x \in \mathbf{R}^n$. This problem comes from unconstrained optimization problems, when g is the gradient mapping of some function $f : \mathbf{R}^n \to \mathbf{R}$, (1.1) is just the first order necessary condition for the unconstrained optimization problem as follows

$$\min \quad f(x) \quad x \in \mathbf{R}^n. \tag{1.2}$$

Among all kinds of methods for solving systems of nonlinear equations, Newton method is one of the most elementary, popular and important methods. One of the advantages of the method is its local quadratic convergence. However, its computational cost is expensive, particularly when the size of the problem is very large, because at each iteration step, the corresponding Newton iteration equation of (1.1) at the kth iteration as follows

$$H_k v = -g_k, \tag{1.3}$$

should be solved exactly. Here $g_k = g(x_k)$, $H_k = H(x_k)$ is the symmetric Jacobian matrix of $g(x)$ at the current iterate x_k. In order to reduce the computational cost of Newton method, Dembo, Eisenstat and Steihaug proposed and studied inexact

☆ The authors gratefully acknowledge the partial supports of the National Natural Science Foundation of China (11371253).

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http://dx.doi.org/10.1016/j.apnum.2016.08.013
0168-9274/

Newton method in [13] which was a generalization of Newton method. The inexact Newton-type method aims to stop the iteration with the relative residual less than a given tolerance. By choosing suitable stopping criteria, we can reduce the total cost of the whole iterations. Evidently, at each iteration step of the inexact Newton method we only need to solve the Newton equation approximately by an efficient iteration solver for systems of linear equations such as the classical splitting methods or the modern Krylov subspace methods. This makes the inexact Newton method more practical and effective than the Newton method in actual applications, because its cost for computing a Newton step may be considerably reduced. Research on inexact-Newton method has widely expanded over the last few decades, and reviews can be found in An [1] and Eisenstat and Walker [15]. The stopping relative residual control guarantees the local convergence of the method [15,16] under the usual assumptions for inexact-Newton's method. However, iterative methods with global convergence behavior are more practical and efficient for solving nonlinear equations and optimization problems. The backtracking procedure or trust-region technique is the typical globalization strategy. With the development of Krylov subspace projection methods, some hybrid globally convergent modifications of inexact-Newton methods have been considered to enhance convergence from arbitrary starting points. A class of methods based on the Arnoldi procedure [3] like FOM/IOM and mainly GMRES [22] is widely used as the inner iteration for inexact-Newton methods [4,9].

We know that if the problem size n is very large, and the Jacobian matrix is sparse, the Krylov subspace iteration method GMRES is a powerful algorithm for computing an inexact Newton step [10]. These combined methods are called inexact Newton–Krylov methods or nonlinear Krylov subspace projection methods. These methods have the virtue of requiring minimum matrix storage and potential matrix-free implementations, resulting in a distinct advantage over direct methods for solving the large Newton equations. Newton-type iteration schemes for solving the nonlinear equations problem using Krylov subspace projection methods as an inner linear solver are considered by many authors including Brown and Saad [7,8]. Their computational results show that these methods can be quite effective for many classes of problems in the context of systems of partial differential equations or ordinary differential equations. An attractive feature of the Newton–Krylov method is that it only requires the action of the Jacobian matrix $H(x)$ on a vector v, and $H(x)v$ can be accurately approximated by a difference quotient of the form

$$H(x)v \approx \frac{g(x+\theta v) - g(x)}{\theta}, \tag{1.4}$$

where θ is a small positive number. The approximation given by (1.4) does not require an explicit knowledge of the Jacobian matrix, which is suitable for large-scale problems. Therefore, these Newton-GMRES type methods could avoid direct computations of the Jacobian matrices, and then results in a matrix-free iteration process [5,18].

Stimulated by the progress in these aspects, the purpose of this paper is to propose an inexact-Newton via GMRES subspace method for solving symmetric nonlinear equations (1.1). We are concerned with an inexact-Newton via GMRES subspace method, where GMRES is used as the inner iteration solver for solving (1.3) approximately. This method is an extension of Newton-GMRES without backtracking techniques designed to solve systems of nonlinear equations. The globalization strategy of the proposed method depends on the property of GMRES iteration and the acceptable rule of search direction. GMRES iteration guarantees that the residual norm $\|H_k v + g_k\|$ of (1.3) is nonincreasing at each iteration. Further, we obtain the conclusion of Lemma 3.2, i.e., $\|H_k v + g_k\| \le \tau \|g_k\|$ for each v generated by the GMRES iteration. Meanwhile, we combine the acceptable rule (2.6) of search direction and the predicted reduction $\|g_k\| - \|H_k v_i + g_k\|$ satisfying the sufficient decrease condition (Lemma 3.3) to obtain a sufficient decrease of the actual reduction on $\|g\|$ at each iteration. Thus, we obtain the global convergence condition which is equal to the condition in [15]. Further, we point out that our method is consistent with efficient matrix-free implementation.

The paper is organized as follows. In Section 2, we review the preliminaries of GMRES iteration and present the main algorithm. The properties of the proposed algorithm are proved in Section 3. The global convergence and the local superlinear convergence rate of the algorithm are established under some reasonable conditions in Section 4. The results of some numerical experiments of the algorithm are reported in Section 5. Finally, in Section 6, we end the paper with a brief conclusion.

2. Preliminaries of GMRES iteration and algorithm description

At each iteration of the inexact-Newton algorithm, we must obtain an approximate solution of the linear system (1.3) i.e.,

$$H_k v = -g_k, \tag{2.1}$$

where $g_k = g(x_k)$, H_k is the symmetric Jacobian matrix of $g(x)$ at x_k and nonsingular. If v_0 is an initial guess for the true solution of (2.1), then letting $v = v_0 + z$, by substituting this relation into (2.1), we obtain the equivalent system after some rearrangement

$$H_k z = -g_k - H_k v_0 = r_0,$$

where r_0 is the initial residual. Define the Krylov subspace $K(H_k, r_0, i)$,