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Spline collocation for fractional weakly singular integro-differential equations ☆

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A B S T R A C T

We consider a class of boundary value problems for linear fractional weakly singular integro-differential equations which involve Caputo-type derivatives. Using an integral equation reformulation of the boundary value problem, we first study the regularity of the exact solution. Based on the obtained regularity properties and spline collocation techniques, the numerical solution of the boundary value problem by suitable non-polynomial approximations is discussed. Optimal global convergence estimates are derived and a super-convergence result for a special choice of grid and collocation parameters is given. A numerical illustration is also presented.

1. Introduction

Differential equations with derivatives of fractional (non-integer) order have recently proved to be valuable tools in the modeling of many physical phenomena [3,4,15,16,22]. Therefore theoretical and numerical analysis of fractional differential equations has been receiving increasing attention by many researchers. For details, including basic theory and some applications, see the monographs [4,8,17,31,34] and review papers [1,5]. Many works are devoted to the analysis and numerical solution of initial or boundary value problems for fractional differential equations. Some recent results concerning fractional initial value problems and fractional boundary value problems can be found in [7,11,13,18,24,25,27,36] and [2,9,12,14,19, 26,28,33,37], respectively. Somewhat less attention has been paid to fractional integro-differential equations [10,20,21,23,29, 30,32]. In particular, very little has been written on solving fractional integro-differential equations with weakly singular kernels [38]. This motivated us in the present paper to focus on constructing effective numerical methods for fractional weakly singular integro-differential equations.

In the present paper we consider a possibility to construct high order numerical schemes for solving boundary value problems for fractional integro-differential equations of the form

$$(D_*^\alpha y)(t) + h(t)y(t) + \int_0^t K(t,s)y(s)ds + \int_0^t \widetilde{K}(t,s)(D_*^\beta y)(s)ds = f(t),\ 0 \le t \le b, \tag{1.1}$$

$$\gamma_0 y(0) + \gamma_1 y(b_1) = \gamma,\ 0 < b_1 \le b,\ \gamma_0, \gamma_1, \gamma \in \mathbb{R} := (-\infty, \infty),\ \gamma_0 + \gamma_1 \ne 0, \tag{1.2}$$

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where D_*^α and D_*^β are the Caputo differential operators of order α and β, respectively. We assume that $0 < \beta < \alpha < 1$, $h, f \in C[0,b]$ and

$$K(t,s) := (t-s)^{-\kappa} K_1(t,s), \quad \widetilde{K}(t,s) := (t-s)^{-\widetilde{\kappa}} \widetilde{K}_1(t,s), \quad (t,s) \in \Delta, \tag{1.3}$$

where $0 \le \kappa < 1$, $0 \le \widetilde{\kappa} < 1$, $K_1, \widetilde{K}_1 \in C(\bar{\Delta})$ and

$$\Delta = \{(t,s) : 0 \le t \le b,\ 0 \le s < t\}, \ \bar{\Delta} = \{(t,s) : 0 \le s \le t \le b\}.$$

By $C[0,b]$ is denoted the Banach space of continuous functions $u : [0,b] \to \mathbb{R}$ with the norm $\|u\|_\infty = \sup\{|u(t)| : 0 \le t \le b\}$. By $C^m(\Omega)$ $(m \ge 0)$ we denote the set of m times continuously differentiable functions on Ω, $C^0(\Omega) = C(\Omega)$. The Caputo differential operator D_*^δ of order $\delta \in (0,1)$ can be defined by formula (see, e.g. [8])

$$(D_*^\delta y)(t) := (D^\delta [y - y(0)])(t), \quad t > 0.$$

Here $D^\delta y$ is the Riemann–Liouville fractional derivative of y:

$$(D^\delta y)(t) := \frac{d}{dt}(J^{1-\delta} y)(t), t > 0, \quad \delta \in (0,1),$$

with J^δ, the Riemann–Liouville integral operator, defined by

$$(J^\delta y)(t) := \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} y(s)\, ds, \quad t > 0, \quad \delta > 0; \quad J^0 := I, \tag{1.4}$$

where I is the identity mapping and Γ is the Euler gamma function.

It is well known (see, e.g. [6]) that J^δ, $\delta > 0$, is linear, bounded and compact as an operator from $L^\infty(0,b)$ into $C[0,b]$, and we have for any $y \in L^\infty(0,b)$ that (see, e.g. [17])

$$J^\delta y \in C[0,b], \quad (J^\delta y)(0) = 0, \quad \delta > 0, \tag{1.5}$$

$$D^\delta J^\eta y = D_*^\delta J^\eta y = J^{\eta-\delta} y, \quad 0 < \delta \le \eta. \tag{1.6}$$

Using an integral equation reformulation of problem (1.1)–(1.2), we first study the existence and regularity of the exact solution. Based on the obtained regularity properties of the exact solution and spline collocation techniques on special non-uniform grids, high order numerical schemes for solving (1.1)–(1.2) are constructed. Our aim is to study the attainable order of the proposed algorithms in a situation where the higher order (usual) derivatives of $h(t)$ and $f(t)$ may be unbounded at $t = 0$. Our approach is based on some ideas of [26]. In particular, the case where (1.1)–(1.2) is an initial value problem ($\gamma_0 \ne 0$, $\gamma_1 = 0$) or a terminal value problem ($\gamma_0 = 0$, $\gamma_1 \ne 0$, see [12,14]) is under consideration.

The rest of the paper is organized as follows. In Section 2 a result about the smoothness of the exact solution to (1.1)–(1.2) is presented (see Theorem 2.1). Later this result will play a key role in the convergence analysis of the proposed algorithms. In Sections 3 and 4 the description and convergence analysis of the proposed numerical schemes are given. Finally, in Section 5 the theoretical results are tested by some numerical experiments.

The main results of the paper are given by Theorems 2.1, 4.1 and 4.2.

2. Existence and smoothness of the solution

In what follows we use an integral equation reformulation of (1.1)–(1.2). Let $y \in C[0,b]$ be an arbitrary function such that $D_*^\alpha y \in C[0,b]$, where $0 < \alpha < 1$. For the solution y of (1.1)–(1.2) we will later show that these assumptions are fulfilled (see Theorem 2.1).

Denote $z := D_*^\alpha y$. Then (see [8,17])

$$y(t) = (J^\alpha z)(t) + c, \tag{2.1}$$

where c is a constant. Due to (1.5) a function of the form (2.1) satisfies the boundary conditions (1.2) if and only if $c(\gamma_0 + \gamma_1) = \gamma - \gamma_1 (J^\alpha z)(b_1)$, that is, if $y(t)$ is determined by formula

$$y(t) = (J^\alpha z)(t) + \frac{\gamma}{\gamma_0 + \gamma_1} - \frac{\gamma_1}{\gamma_0 + \gamma_1}(J^\alpha z)(b_1), \quad 0 \le t \le b. \tag{2.2}$$

Let $y \in C[0,b]$ be a solution of problem (1.1)–(1.2) such that $D_*^\alpha y \in C[0,b]$. Substituting (2.2) into (1.1) and using (1.6), we obtain that $z = D_*^\alpha y$ is a solution of an integral equation of the form

$$z = Tz + g, \tag{2.3}$$

where