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Adaptive model order reduction for the Jacobian calculation in inverse multi-frequency problem for Maxwell's equations

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ABSTRACT

This work develops a model order reduction method for a numerical solution of an inverse multi-frequency eddy current problem using a rational interpolation of the transfer function in the complex plane. We use the Pade interpolation in the complex frequency plane; this allows us to speed up the calculation of the frequency-dependent Jacobian in the inversion procedure without loosing accuracy. Interpolating frequencies are chosen adaptively to reduce the maximal approximation error. We use the error indicator that is equivalent to a seminorm of the residual. The efficiency of the developed approach is demonstrated by applying it to the inverse magnetotelluric problem, which is a geophysical electromagnetic remote sensing method used in mineral, geothermal, and groundwater exploration. In this application, the transfer function values are needed for shifts in a purely imaginary interval. Thus we consider the interpolating shifts in the same interval as well as in a purely real interval, containing the spectrum of the operator. Numerical tests show an excellent performance of the proposed methods characterized by a significant reduction of computational time without loss of accuracy of the calculated Jacobian.

1. Introduction

Model order reduction (MOR) is a powerful technique for reduction of the dimensionality of a problem. It is especially efficient when the low dimensional subspace is generated using rational Krylov subspaces. This approach has become popular recently and has been used in a variety of contexts [33,1,4,14,28,32,31,2,11,13,15,16]. The efficiency of the method is amplified significantly by an optimal selection of the shifts for generating the rational Krylov subspace. For a particular matrix with uniform spectrum, the problem of selection of the optimal shifts has been investigated in [27,34,36]. However, this selection is not optimal for matrices with non-uniform spectrum. An excellent review of this topic is presented in [17].

A different approach, in which the shifts are added one by one in a greedy fashion, was developed in [7,5]. In this approach, the shifts are adapted to the spectrum of the matrix. The authors of [7] consider an adaptive choice of shifts for the approximation of the transfer function using rational Krylov subspaces, with an application to a time-domain electromagnetic geophysical forward problem. This adaptive choice of shifts was later generalized to non-symmetric matrices

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[8] and to approximation of matrix functions other than the transfer function [18]. Numerical simulations in [7,18] show that the adaptive approach gives better results than the choice of the shifts that do not depend on the spectrum of the operator. Moreover, these numerical results demonstrate that the number of required shifts is not increased with the size of the system, and it is not strongly dependent on the spectrum. Those results encourage us to pursue an adaptive choice of the shifts in the current work.

The work of [7] and the rational Krylov subspace approach was extended to the inverse problem [9,38], where the model order reduction may be viewed as constraint on the set of admissible conductivity models or as a way to calculate a good approximation of the Jacobian with a greatly reduced computational time. For many inverse problems solved using a method (like Gauss–Newton) requiring a Jacobian, significant and sometimes dominant computational cost is related to its calculation. In the current paper, we use the model order reduction through rational Krylov subspaces only to speedup the calculations of the transfer function used in the evaluation of the Jacobian, with no loss of accuracy. The proposed method can be used with any standard inversion approach that requires a calculation of the Jacobian.

We develop the adaptive choice of optimal shifts method with an application to magnetotellurics (MT), which is a frequency domain electromagnetic remote-sensing geophysical method used in mineral, geothermal, and groundwater exploration. Numerically simulating the scattering of EM waves from complex three-dimensional (3D) structure is a computationally demanding problem [6,12]. In particular, the scattering response usually needs to be calculated over a broad frequency range. Typically this may be five orders of magnitude or more with frequency sampling of 5–10 base points per decade. A considerable savings in computational time could result if an accurate method of interpolation of responses across a coarser selection of base points can be achieved.

In this case, for the forward problem, the transfer function $\tilde{h}(s) = (\tilde{A} + s\tilde{B})^{-1}\tilde{b}$ has a complex valued right hand side $\tilde{b}$ dependent on frequency; the model order reduction method that allows to treat such problem is developed in [24]. In the case of the calculation of the Jacobian, the rhs $\tilde{b}$ is not dependent on frequency and is real valued. In our application, $\tilde{B}$ is not the identity matrix. We show that it is related to the case of $\tilde{B} = I$ through a scaling $\tilde{B}^{1/2}$. The scaling is useful to simplify the analysis, but its calculation is not required in the approximation procedure. We also present a proof of the transfer function approximation error formula. The proof is different than the one in [22].

The transfer function values are required in a purely imaginary interval, so we consider the interpolating shifts in the same imaginary interval. We also consider real shifts, following the suggestion of [7]. The shifts are chosen to reduce the maximal error of interpolation and as the true error is unknown, an error indicator is needed. We consider error indicator suggested by [7] and we show that it is equivalent to any seminorm of the residual. The speedup of the algorithm with the model order reduction is higher when the number of frequencies considered in MT survey increases. In our numerical tests, the speedup of calculation of the Jacobian for 30 frequencies, is 4 times.

The paper is organized as follows. In section 2 we present the formulation of the forward magnetotelluric problem and Jacobian required in the inversion procedure. We explain how the approximation of the transfer function may be used to speedup the calculations. Next we show the theory of the approximation of the transfer function using rational Krylov subspaces. In section 3 we present the error indicator function and present two algorithms based on them. We also give details of the numerical implementation and discuss a possibility of using quadruple precision for some of the non-computationally demanding calculations. In section 4 we show results of numerical tests for a 3D magnetotelluric model with non-constant conductivity structure and with a hill and a valley in topography.

2. Theory

2.1. Inverse magnetotelluric problem

The forward and inverse magnetotelluric (MT) problem is described in detail in [25,26]. We consider a domain Ω that includes both the air and the earth's subsurface. The earth's surface is allowed to have a non-flat topography. In order to calculate the MT response due to an arbitrary 3D conductivity structure $\sigma > 0$ we consider the edge finite element discretization of the equation for the secondary electric field E. Though the numerical tests presented are done using lowest order edge hexahedral discretization, all the methods may be applied to other discretizations, such as the tetrahedral mesh, higher order edge elements as well as the finite difference method.

The solution space for the unknown electric field is defined as

$$\mathcal{H}_0(\nabla\times, \Omega) = \left\{F : \Omega \to \mathbb{C}^3 : \int_\Omega \left(|F|^2 + |\nabla \times F|^2\right) < \infty,\ \ n \times F|_{\partial\Omega} = 0\right\} \tag{1}$$

Consider Maxwell's equations in the frequency domain for a low angular frequency ω. We denote the magnetic permeability by μ and the permittivity by ϵ. The term $\mathrm{i}\omega\epsilon$, related to the displacement current is neglected. The equation for the secondary field $E \in \mathcal{H}_0(\nabla\times)$ is

$$\int_\Omega \frac{1}{\mu}\nabla \times E \cdot \nabla \times F + \mathrm{i}\omega \int_\Omega \sigma E \cdot F = \int_\Omega -\mathrm{i}\omega(\sigma - \sigma^p)E^p \cdot F \tag{2}$$

for all test functions $F \in \mathcal{H}_0(\nabla\times)$. The source term in (2) depends on primary electric field E^p, which is a plane wave traveling in the medium of the primary conductivity σ_p of a 1D earth. The conductivity σ is an arbitrary nonnegative function of position in the 3D domain, satisfying the assumption that $\sigma \approx \sigma_p$ close to the domain boundaries.